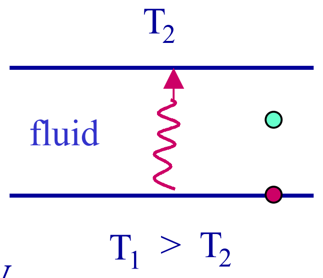
Institute for Theoretical Physics
Eötvös University
E-mail: racz@poe.elte.hu
Homepage: poe.elte.hu/~racz

Distribution Functions for Nonequilibrium Fluctuations: A Picture Gallery

by Zoltán Rácz

Question: What is the probability distribution function (PDF) of a macroscopic quantity in a far from equilibrium system?



Recent attempts: Nonextensive statistics (Tsallis)
Superstatistics (Beck - Cohen)

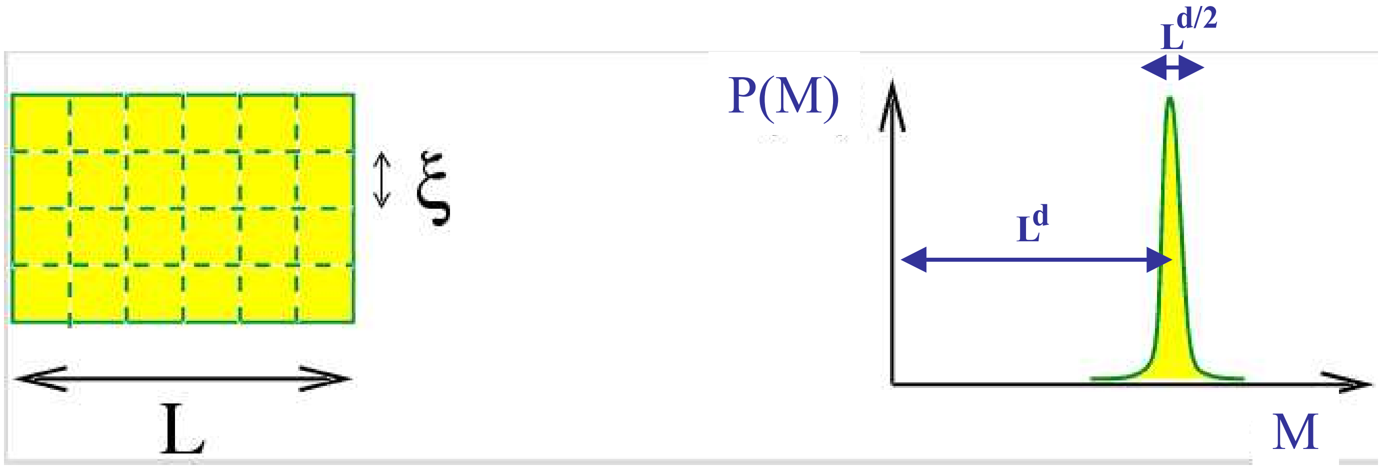
Simpler question: Do power-law correlations (effective criticality) imply
(1) nongaussian PDF-s,
(2) universal forms according to universality classes?

Aims: (1) Construct a picture gallery of scaling functions (PDF-s).
(2) Identify underlying nonequilibrium features by comparing PDF-s

Results: picture gallery - surface growth experiments •
scalability of parallel algorithms,
turbulence and surface fluctuations,
upper critical dimension of KPZ equation •
1/f noise and extreme statistics •

T. Antal
M. Plischke
M. Droz
G. Gyorgyi
P. Holdsworth
R.K.P. Zia
F. van Wijland

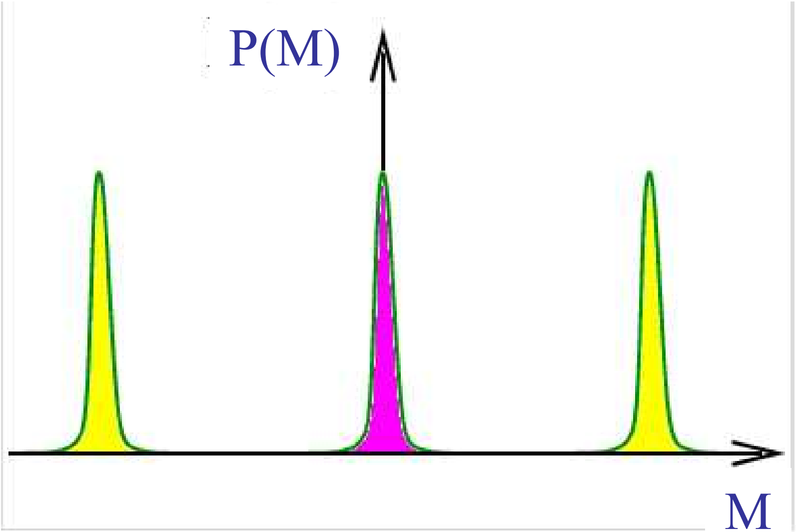
Gaussian and nongaussian distributions I



Extensive quantity in a noncritical system ($\xi \ll L$)

central limit theorem →

Small Gaussian fluctuations around the mean



Example: Ising model above and below Tc.

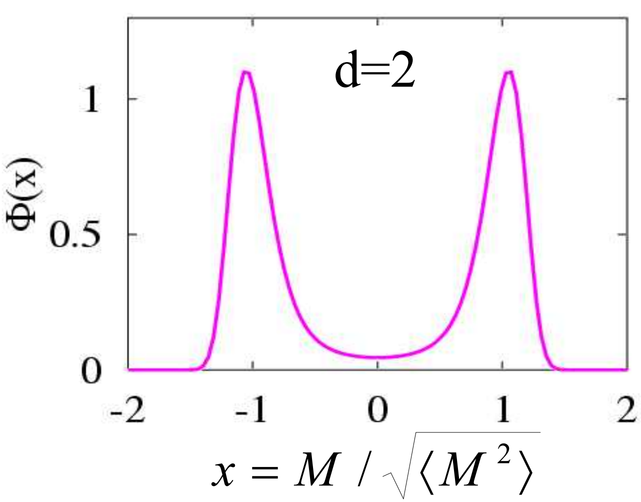
Gaussian and nongaussian distributions II

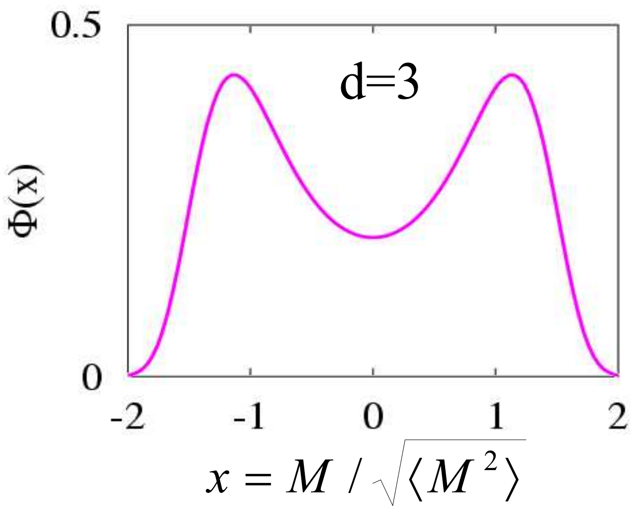
Extensive quantity in a critical system ($\xi \sim L$) —no central limit theorem→ nongaussian fluctuations around the mean

Example: Ising model at Tc - emergence of universal scaling functions

Scaling variable: $x = M / \sqrt{\langle M^2 \rangle}$

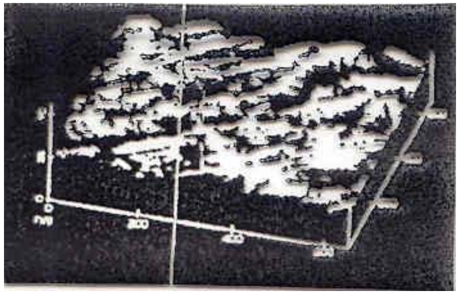
Scaling func.: $\Phi(x) = \sqrt{\langle M^2 \rangle}\, P(M)$



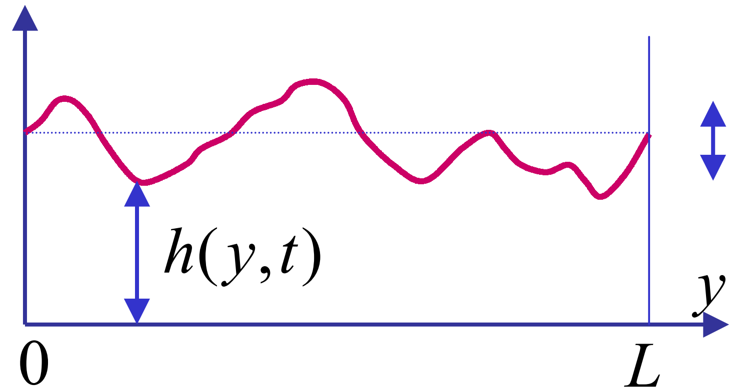


Boundary condition dependence - a characteristics of nonequilibrium systems

Nongaussianity - width distribution of interfaces



Ag on glass



$$\sqrt{\langle w_2 \rangle} \sim L^{\chi}$$

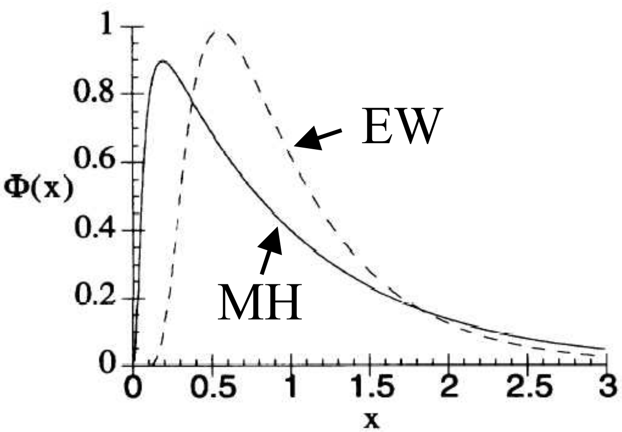
width

Stationary distribution:

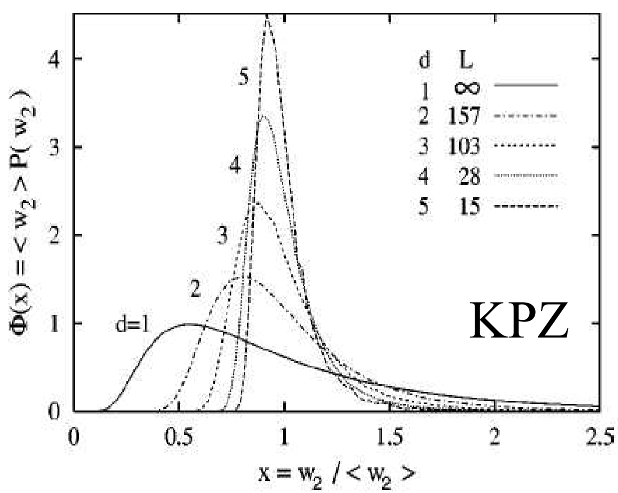
$$\sqrt{\langle w_2 \rangle_L}\; P(w_2) = \Phi\left(\frac{w_2}{\langle w_2 \rangle_L}\right) = \Phi(x)$$

$$w_2(L,t) = \frac{1}{L}\int_0^L dy\left[h(y,t) - \bar{h}\right]^2$$

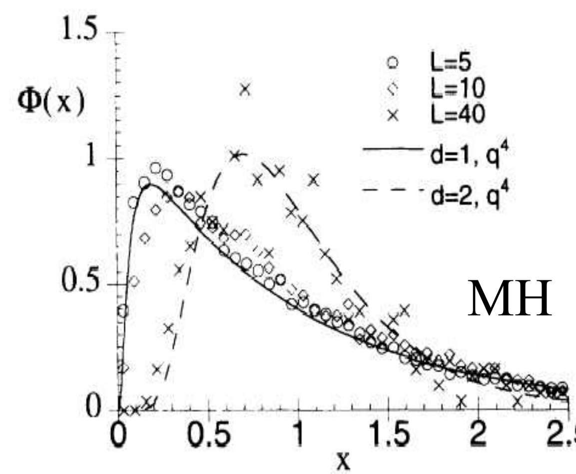
Picture gallery



PRE50, 639, 3589 (1994)



PRE65, 026136 (2002)

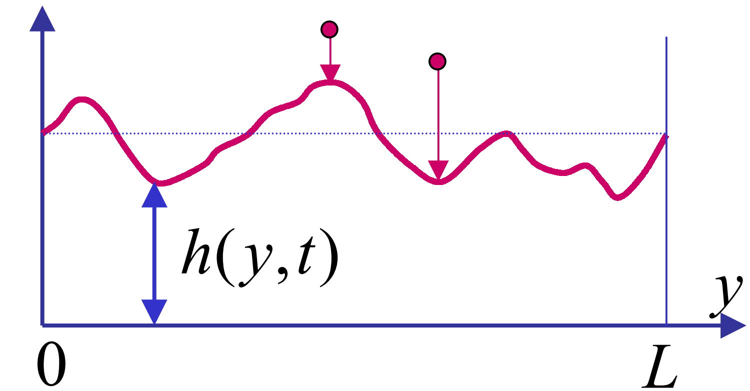


PRE50, 3530 (1994)

Nongaussian distributions - Edwards-Wilkinson (EW) interface

$$\partial_t h = \sigma\, \Delta h + \eta$$

surface tension driven dynamics



Stationary state: Random Walk

$$w_2 = \frac{1}{L}\int_0^L dy (h - \bar{h})^2 = \sum_k |h_k|^2 \propto L$$

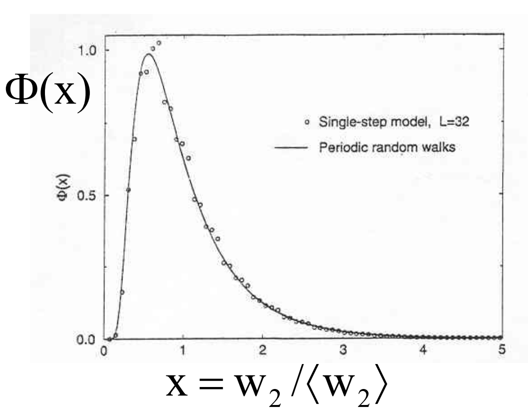
sum of independent variables, diverging fluctuations

$$P[h] \sim e^{-\sigma \int_0^L (\nabla h)^2 dy}$$

Fourier modes:

$$\sim e^{-\sigma L \sum_k k^2 |h_k|^2} = \prod_k e^{-\sigma L k^2 |h_k|^2}$$

independent modes



Reason for the failing of the central limit theorem

$$\langle |h_k|^2 \rangle \sim \frac{1}{k^2}$$

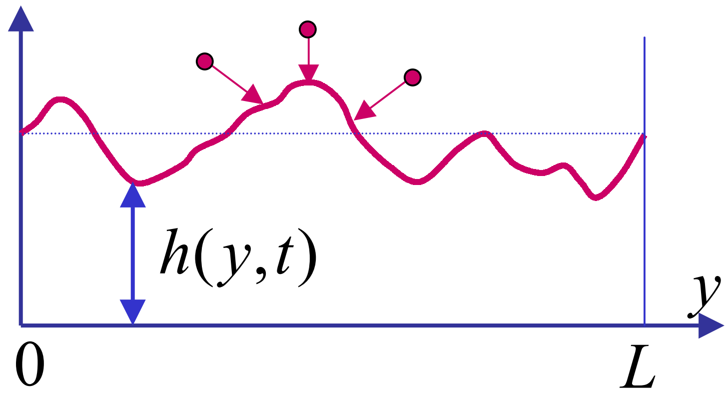
nonidentical, singular fluctuations

Nongaussian distributions -Kardar-Parisi-Zhang (KPZ) interface

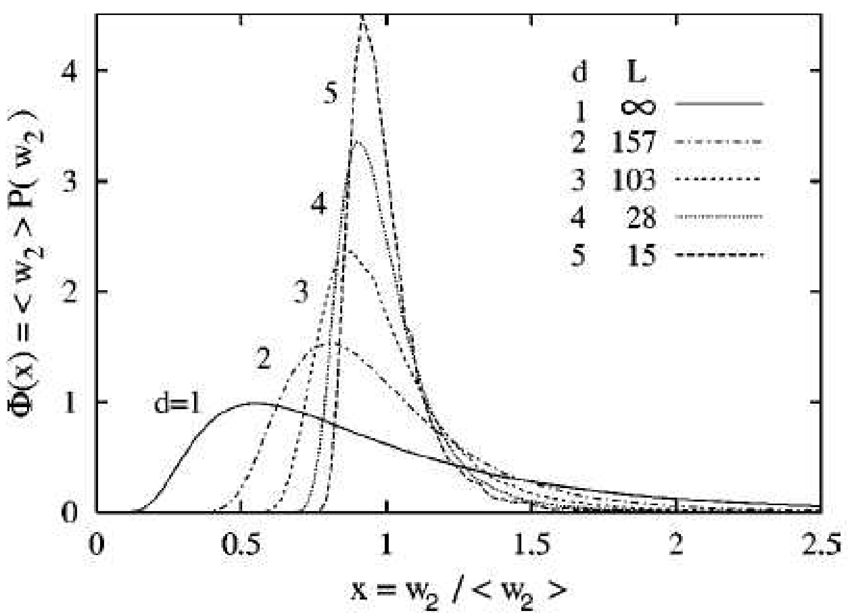
PRE65, 026136 (2002)

$$\partial_t h = \sigma\, \Delta h + \lambda (\nabla h)^2 + \eta$$

attachment: growth along the normal
+ surface tension effects



Q: Is there an upper critical dimension?



$$\langle |h_k|^2 \rangle \sim \frac{1}{k^{2+\zeta}}$$

Nonidentical, singular,
nongaussian fluctuations

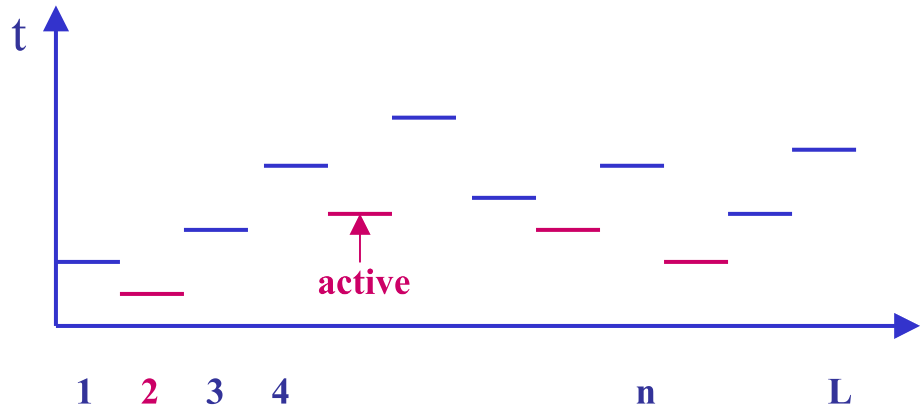
No parameters to fit

Scalability of Parallel Algorithms (fluctuating time-horizon)

G. Korniss et al. PRL84, 1351 (2000)

Local time of the computer

Only local minima can compute

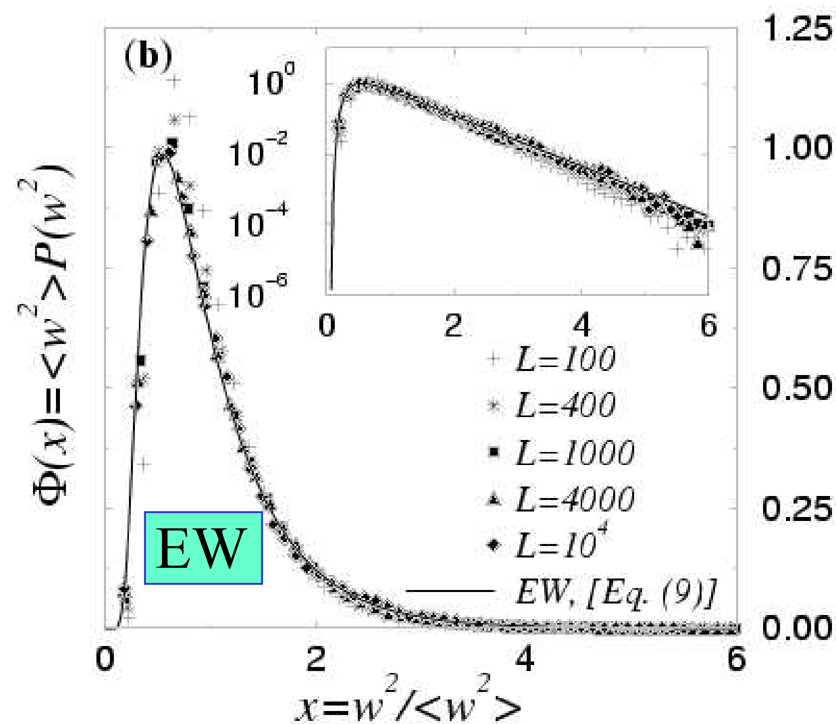


Events:
Poisson arrivals

How does the number of active computers grow with L?

$$N_{active} \sim L^{\alpha}$$

$$\alpha = 1$$



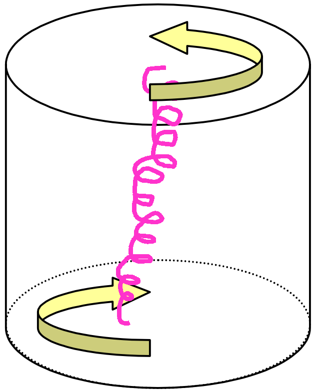
Turbulence and the d=2 EW model

S. Bramwell et al. Nature 396, 552 (1998)

Experiment

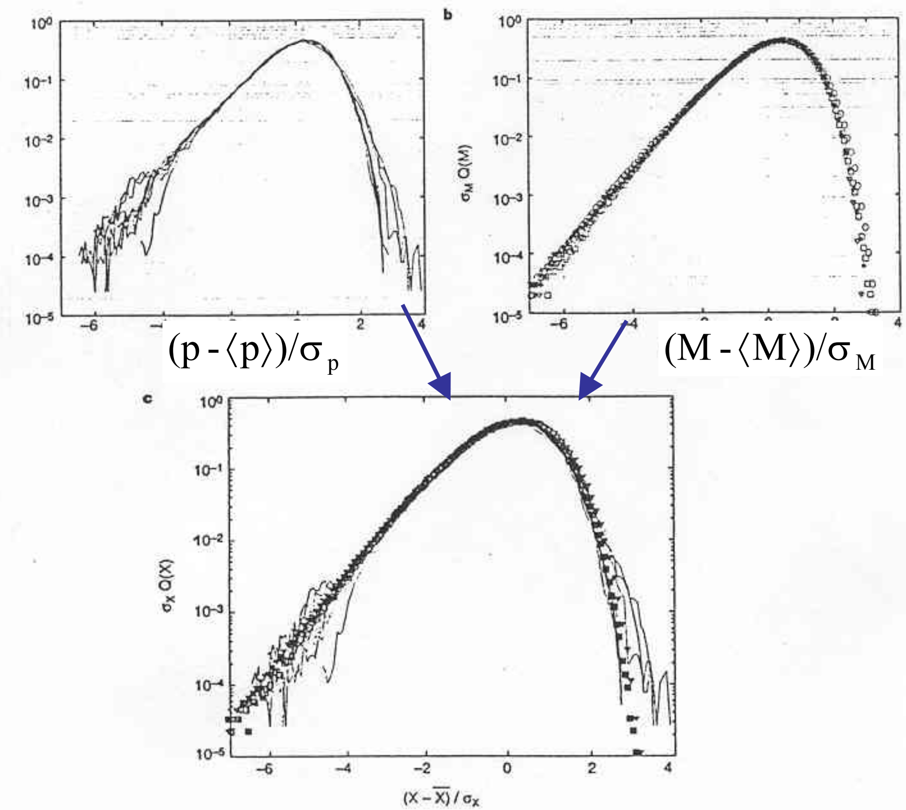
Critical

$$\omega = \omega_c$$



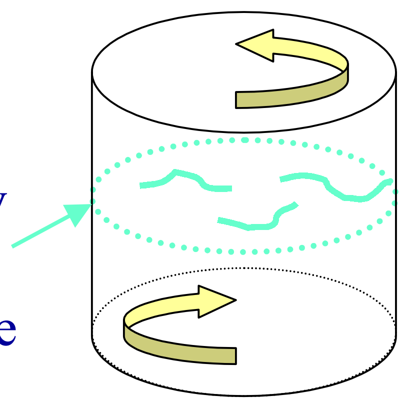
Distribution of energy dissipation

Distribution of d=2 XY magnetization below Tc. (finite-size)



Aji & Goldenfeld, PRL86, 1007 (2001)

dissipation is mainly on the fluctuations of the shear pancake

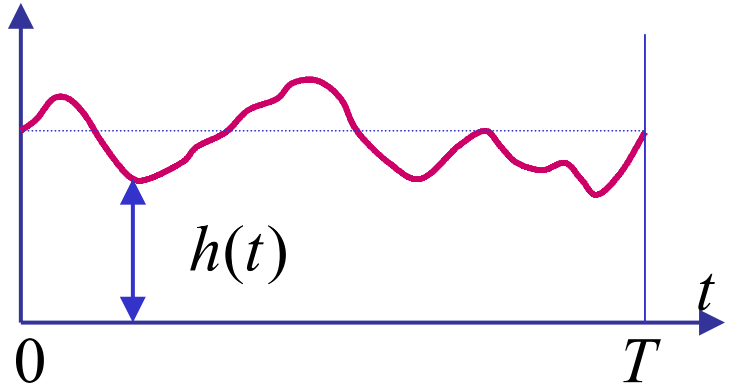


$$d = 2 \ \ EW \equiv XY \ \ T < T_c$$

Possible connection to extreme statistics?

Width distributions for $1/f^{\alpha}$ signals

PRE65, 046140 (2002)



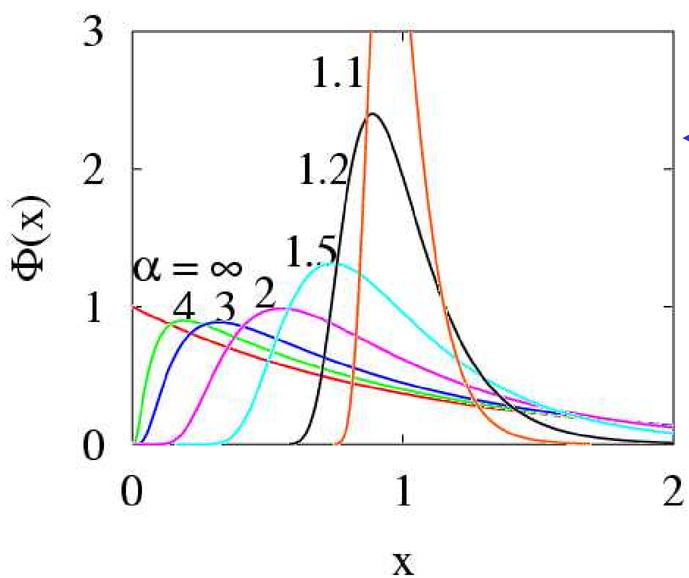
Stationary distribution for Fourier modes

$$\mathbf{P}[h_n] \sim e^{-\sigma \sum_n |n|^{\alpha} |h_n|^2}$$

$$w_2 = \sum_n |h_n|^2$$

Integrated power spectrum

Question: Is there an α for which extreme statistics distribution emerges?



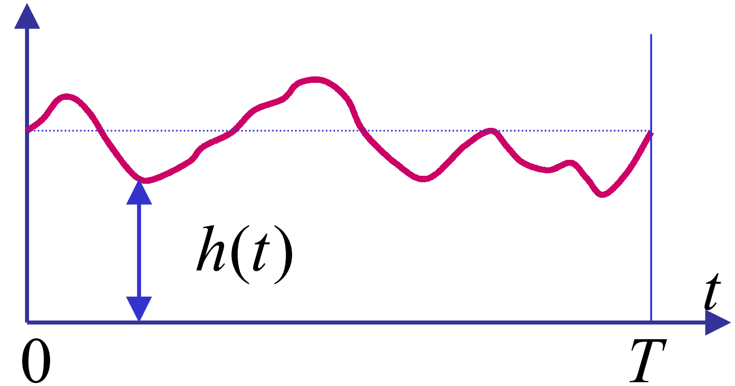
$$\sqrt{\langle w_2 \rangle_T}\, P(w_2) = \Phi_{\alpha}\left(x = \frac{w_2}{\langle w_2 \rangle_T} \right)$$

$$\alpha \to 1 \qquad \Phi_{\alpha}(x) \to \delta(x)$$

Does it have and internal structure?

Extreme statistics for $\alpha = 1$.

PRL87, 240601 (2001)



$\alpha \to 1$ new scaling variable

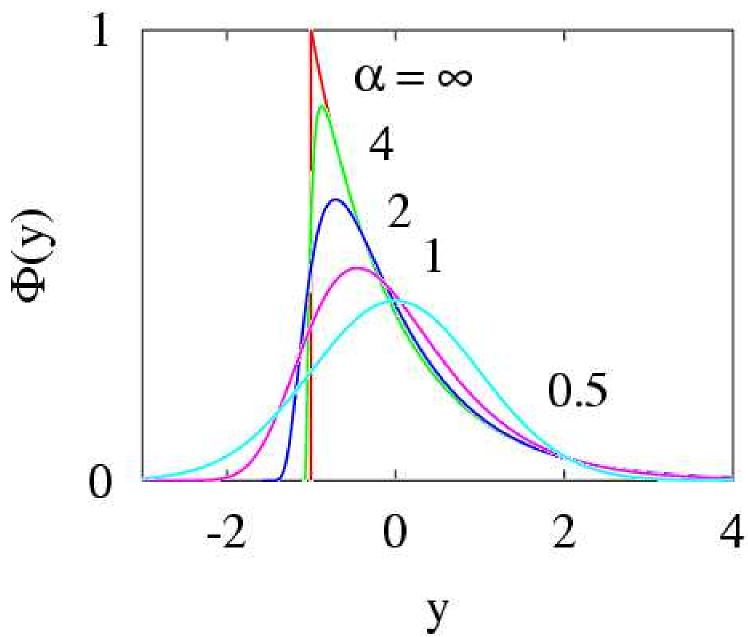
$$y = \frac{w_2 - \langle w_2 \rangle_T}{\sigma_T}$$

$$\sigma_T = \sqrt{\langle w_2^2 \rangle_T - \langle w_2 \rangle_T^2}$$

$$w_2 = \sum_n |h_n|^2$$

integrated power spectrum

$$\sigma_T P(w_2) = \Phi_\alpha(y)$$



$\alpha = 1$

$$\Phi_1(y) = e^{-y-e^{-y}}$$

Fisher-Tippett-Gumbel extreme value distribution

$\alpha \le 1/2$

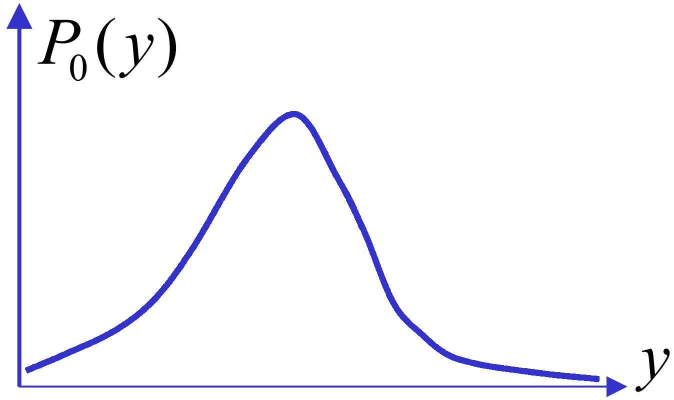
$$\Phi_\alpha(y) \sim e^{-y^2}$$

$\langle |h_n|^2 \rangle \sim n^{-\alpha}$

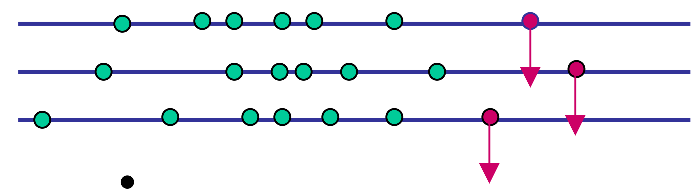
Central limit theorem is restored for $\alpha \le 1/2$

Extreme value statistics

N numbers are drawn from a distribution

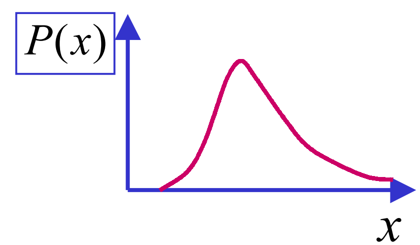


Question: What is the limiting $(N \to \infty)$ distribution of the largest number?



Three types depending on $P_0(y \to \infty) = ?$

$$\rho(x) = \int_{-\infty}^{x} dy\, P(y)$$



I. Faster than any power law (Fisher -Tippet - Gumbel):

$$\rho(x) = e^{-e^{-x}}$$

II. Power law (Fisher-Tippett-Frechet)

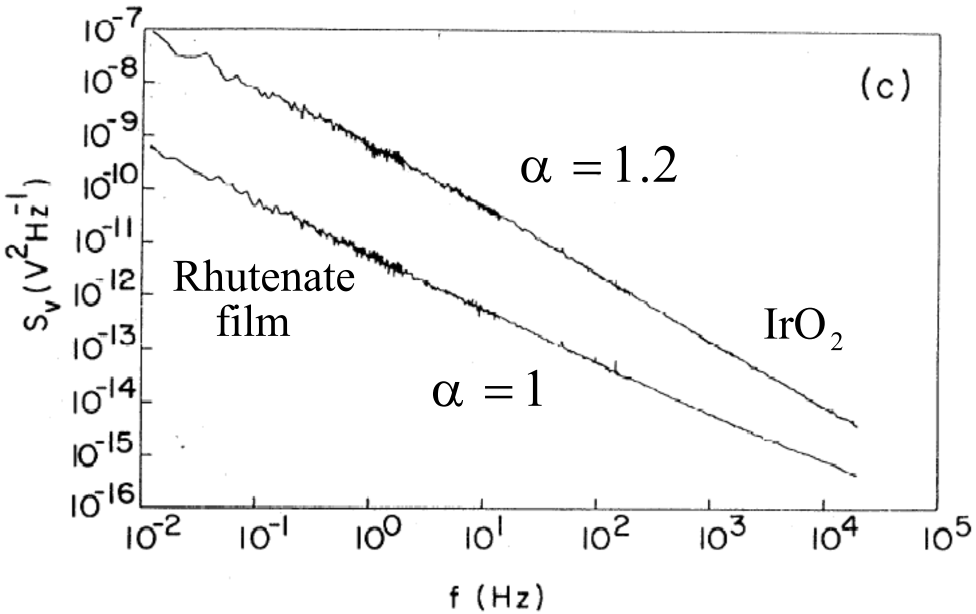
$$\rho(x) = \begin{cases} 0 & x \le 0 \\ e^{-x^{-\alpha}} & x > 0 \end{cases}$$

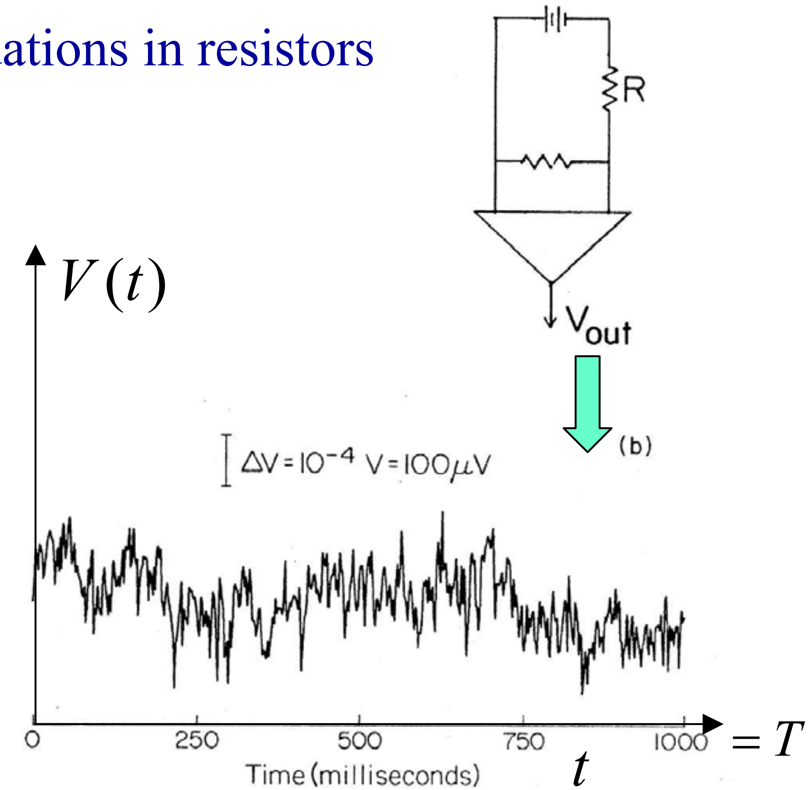
III. Finite cutoff (Weibull)

$$\rho(x) = \begin{cases} e^{-(-x)^{\alpha}} & x \le 0 \\ 1 & x > 0 \end{cases}$$

Where do the power laws come from?

History - examples 1/f noise - voltage fluctuations in resistors





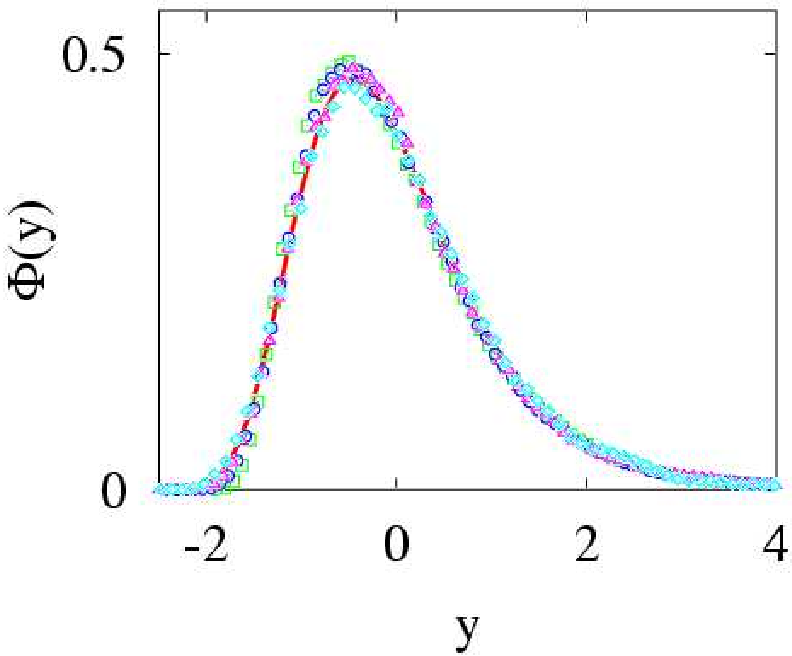
$$S(f) \sim \langle |V_f|^2 \rangle \sim 1/f^{\alpha}$$

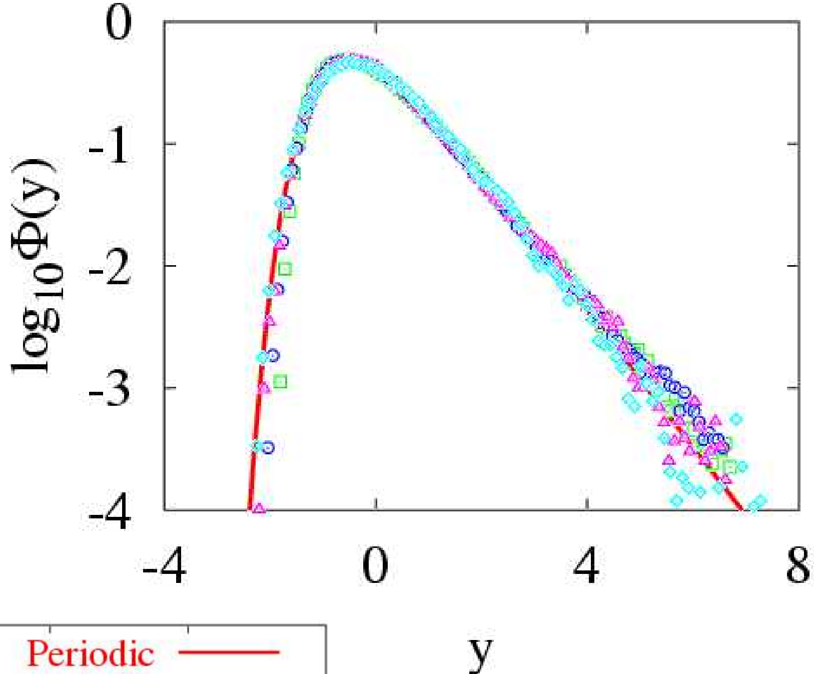
$$V_f = \tfrac{1}{T} \sum_x e^{2\pi i f t} V(t)$$

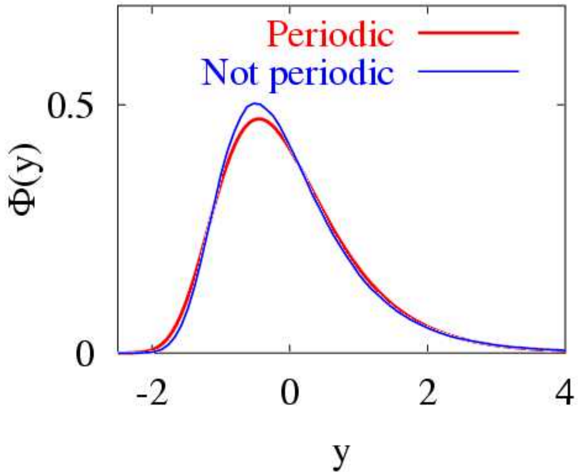
M.B. Weissman, Rev.Mod.Phys.60, 537(1988)

Extreme value statistics - resistivity fluctuations

Experimental data on GaAs films: A.V. Yakimov and F.N. Hooge

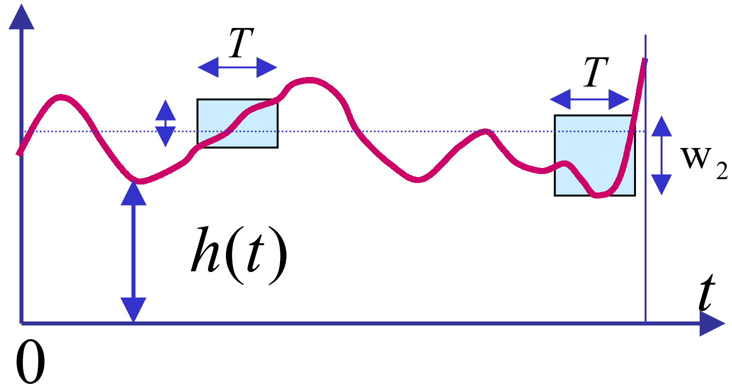






Boundary condition effects

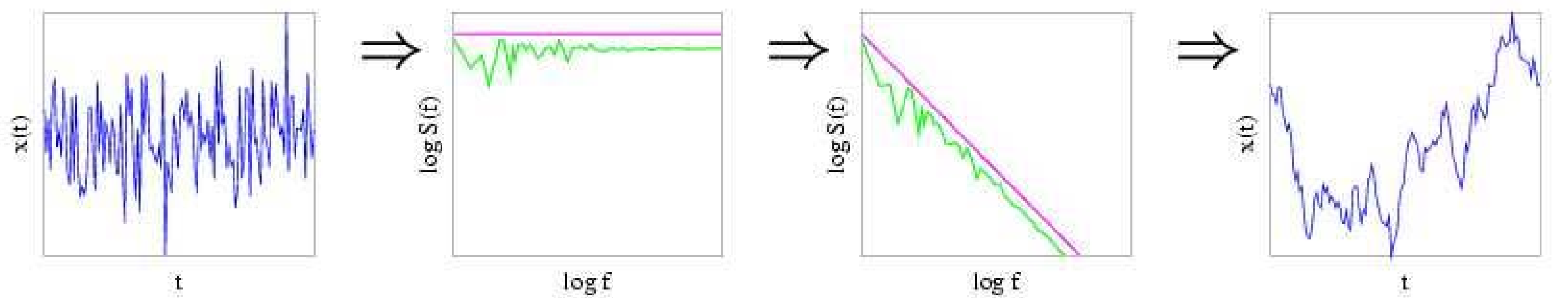
Nonperiodic signals



$\Rightarrow$ $P(w_2)$

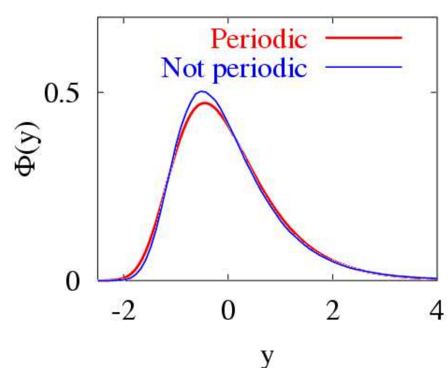
Exact results for $\alpha = 0.5, 2, \infty$

Numerics:



Window b.c.:

Small difference for $\alpha = 1$

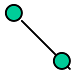


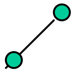
The difference increases with increasing α.

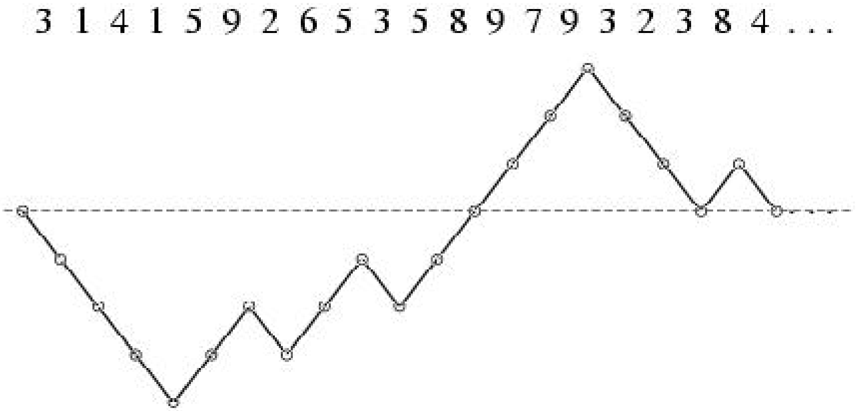
Example with $\alpha = 2$ $\Rightarrow$

Randomness of the digits of π

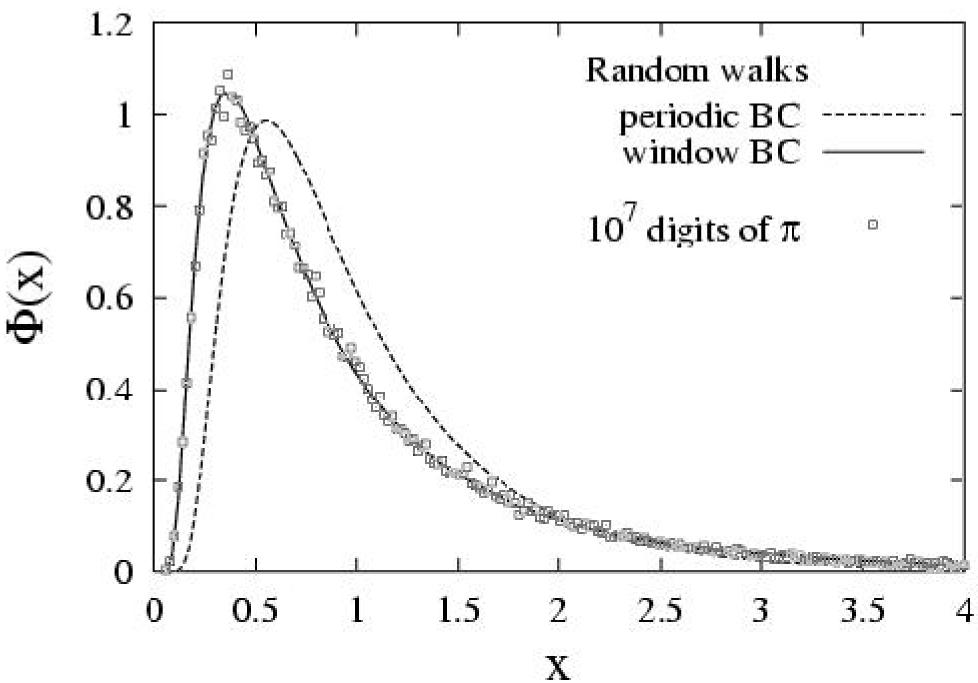
Mapping of π onto a surface


0 … 4


5 … 9

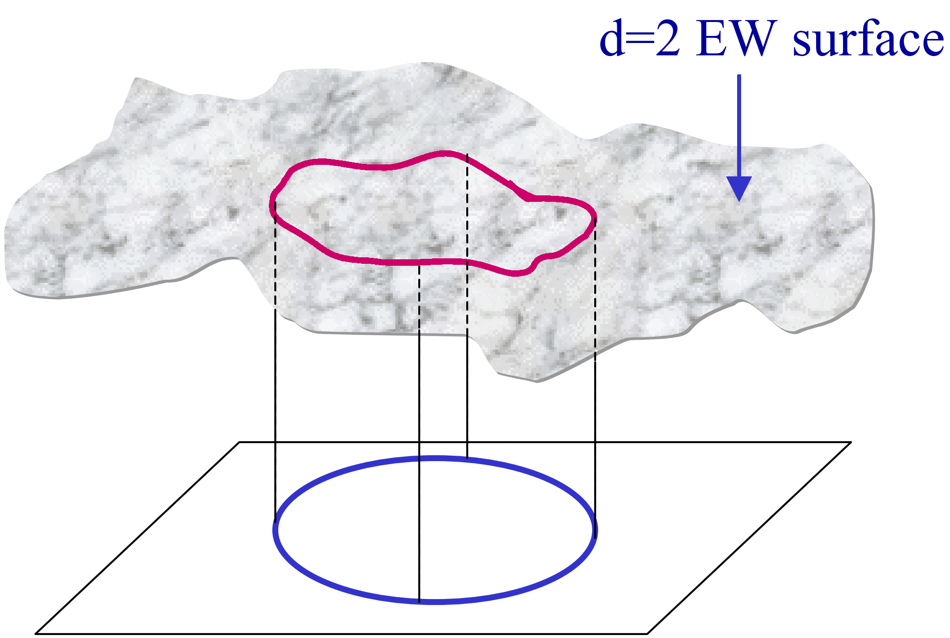


Roughness-distribution of π



Observing the right boundary conditions is clearly important.

Periodic 1/f signal



The fluctuations of red line display a perfect 1/f power spectrum.

Spin Chains with Energy Flux

Transverse Ising model

$$\hat{H}_I = -\sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

Driving the energy flux

$$\hat{H} = \hat{H}_I - \lambda J_E$$