Full Counting Statistics in a Propagating Quantum Front and Random Matrix Spectra

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One-dimensional free fermions are studied with emphasis on propagating fronts emerging from a step initial condition. The probability distribution of the number of particles at the edge of the front is determined exactly. It is found that the full counting statistics coincide with the eigenvalue statistics of the edge spectrum of matrices from the Gaussian unitary ensemble. The correspondence established between the random matrix eigenvalues and the particle positions yields the order statistics of the rightmost particles in the front and, furthermore, it implies their subdiffusive spreading.

The system under consideration is an infinite chain of one-dimensional free fermions described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{m=-\infty}^{\infty} (c^\dagger_m c_{m+1} + c^\dagger_{m+1} c_m),$$

where $c^\dagger_m$ is a fermionic creation operator at site $m$. Initially, all the sites with $m \leq 0$ are filled while those with $m > 0$ are empty and thus the correlations at time $t = 0$ are given by

$$\langle c_m c_n \rangle = \begin{cases} \delta_{mn} & m, n \leq 0 \\ 0 & \text{else} \end{cases}.$$

Since $H$ is diagonalized by a Fourier transform with a single-particle spectrum $\omega_q = -\cos q$, the time evolution of the Fermi operators $c_m(t)$ can be obtained in terms of Bessel functions $J_m(t)$ as

$$c_m(t) = \sum_{j=-\infty}^{\infty} e^{ijm} J_{j-m}(t) c_j.$$

The step function in the density $n_m(t)$ spreads out ballistically and for $m > 0$ is given by [16]

$$n_m(t) = \langle c^\dagger_m(t)c_m(t) \rangle = \frac{1}{2} \left[ 1 - J_0^2(t) - \sum_{k=1}^{m-1} J_k^2(t) \right].$$
with snapshots of the density profile shown on Fig. 1. The curves for different times collapse in the bulk by choosing the scaling variable $m/t$, and the emerging profile can be understood by semiclassical arguments [15]. However, there is a nontrivial staircase structure emerging around the edge of the front, shown by the shaded areas in Fig. 1. The size of the edge region was found to scale with $t^{1/3}$ and it was argued that each step contains one particle [17]. Since the origin of this behavior cannot be captured by a semiclassical treatment, a more detailed understanding will be obtained by looking at the FCS which includes all the higher order correlations of the particle number.

The FCS is defined through the generating function

$$\chi(\lambda, t) = \langle \exp[i\lambda N_A(t)] \rangle,$$

where $N_A(t) = \sum_{m \in A} c_m^\dagger(t)c_m(t)$ is the particle number operator in subsystem $A$ at time $t$. The subsystem will be chosen to include only the edge of the front and thus $A$ itself is time dependent. The generating function can be written as a determinant [13,14]

$$\chi(\lambda, t) = \det[1 + (e^{i\lambda} - 1)C(t)],$$

where $C(t)$ is the reduced correlation matrix at time $t$ with matrix elements $C_{mn}(t) = \langle c_m^\dagger(t)c_n(t) \rangle$ restricted to the subsystem, $m, n \in A$, while $I$ is the identity matrix on the same interval. For the step initial condition one obtains the matrix elements as [16]

$$C_{mn}(t) = \frac{i^{n-m}t}{2(m-n)}[J_{m-1}(t)J_n(t) - J_m(t)J_{n-1}(t)].$$

For convenience, we define $\tilde{C}_{mn}(t)$ by dropping the phase factor $i^{n-m}$ in Eq. (7) which corresponds to a simple unitary transformation of the matrix $C(t)$ and thus leaves $\chi(\lambda, t)$ invariant. Note that $\tilde{C}(t)$ has exactly the same form as the ground-state correlation matrix of a free fermion chain with a gradient chemical potential [22]. Thus our results can be directly applied to the static interface problem as well.

In order to explore the edge region of the front which scales as $t^{1/3}$, we introduce scaling variables $x$ and $y$ through $m = t + 2^{-1/3}t^{1/3}x$ and $n = t + 2^{-1/3}t^{1/3}y$. We also use the identity $J_{m-1}(t) = J_m(t) + \frac{t}{2}J_m(t)$ to rewrite $\tilde{C}_{mn}(t)$ in terms of Bessel functions and their time derivatives as

$$\frac{t}{2(m-n)}[J_m(t)J_n(t) - J_m(t)J_{n-1}(t)] + \frac{1}{2}J_m(t)J_n(t).$$

In the asymptotic regime $t \to \infty$, one has the following forms [23] for the Bessel function and its time derivative $J_m(t) = 2^{1/3}t^{-1/3}A_i(x)$, $J_m(t) = -2^{1/3}t^{-2/3}A_i'(x)$, and the other pair of equations is obtained by interchanging $m \to n$ and $x \to y$. Substituting into Eq. (8) one arrives at

$$\tilde{C}_{mn}(t) = 2^{1/3}t^{-1/3}K(x,y) + 2^{-1/3}t^{-2/3}A_i(x)A_i(y),$$

where

$$K(x,y) = \frac{A_i(x)A_i'(y) - A_i'(x)A_i(y)}{x-y}.$$  

The factor $2^{1/3}t^{-1/3}$ in the first term of Eq. (10) accounts for the change of variables while the other term vanishes in the scaling limit. Therefore, the reduced correlation matrix is turned into an integral operator and the generating function is given by the Fredholm determinant

$$\chi(\lambda, s) = \det[1 + (e^{i\lambda} - 1)K],$$

where $s$ denotes the scaled left endpoint of the domain $A = (s, \infty)$ over which the kernel $K(x, y)$ is defined.

The Airy kernel, Eq. (11), is well known in the theory of random matrices from the Gaussian unitary ensemble (GUE). Namely, it appears in the level spacing distribution at the edge of the spectrum [24,25]. The bulk eigenvalue density of $N \times N$ GUE matrices is described by the so-called Wigner semicircle with the endpoints at $\pm \sqrt{2N}$. However, in order to capture the fine structure of the edge of the spectrum one has to magnify it by choosing the scaling variable $\sqrt{2N} + 2^{-1/2}N^{-1/6}x$. Then the probability that exactly $n$ eigenvalues lie in the interval $(s, \infty)$ is given by the expression [24,25]

$$E(n, s) = \frac{(-1)^n}{n!} \frac{d^n}{dz^n} \det(1 - zK) \bigg|_{z=1}.$$  

We can easily prove now that the eigenvalue statistics, Eq. (13), at the edge of the GUE spectrum is identical to that of the quantum front with $n$ being the number of particles. Indeed, the generating function of Eq. (13) is obtained by a Fourier transform and can be written as
\[ \chi(\lambda, s) = \exp\left( -e^{i\lambda} \frac{d}{dz} \right) \det(1 - zK) \bigg|_{z=1}. \]  

Here, one can notice the operator of the translation which acts as \( \exp(a \frac{d}{dz}) f(z) = f(z + a) \) on any analytical function. Translating the argument and setting \( z = 1 \) afterwards, one immediately recovers the previous result, Eq. (12). We have thus mapped the FCS of the quantum front to the well studied problem of GUE edge eigenvalue statistics. One should point out, however, that while the edge region of the front widens with time, the edge of the random matrix spectrum shrinks with the matrix size and thus the correspondence is valid only after proper rescaling.

As an application of the mapping, we revisit the question of interpretation of the edge density profile in terms of single particles [17], and we also obtain the order statistics of the particles. In the random matrix language, the probability density of the \( n \)th largest eigenvalue for GUE random matrices is given by [24]

\[ F(n, x) = \sum_{k=0}^{\infty} \frac{dE(k, x)}{dx} . \]  

In addition, one has the following sum rules [25]:

\[ \sum_{k=0}^{\infty} E(k, x) = 1, \quad \sum_{k=0}^{\infty} kE(k, x) = \text{Tr}K \]  

with obvious probabilistic interpretations. Using the above sum rules, one arrives at the scaled density of eigenvalues as a sum of single eigenvalue densities

\[ \sum_{n=1}^{\infty} F(n, x) = -\sum_{k=0}^{\infty} \frac{dE(k, x)}{dx} = \rho(x), \]  

where

\[ \rho(x) = K(x, x) = [\text{Ai}'(x)]^2 - x \text{Ai}^2(x) \]  

is obtained from Eq. (11) by taking the limit \( x \to y \).

The results, Eqs. (15) and (17), have clear interpretations in the particle picture. Namely, the probability distribution of the \( n \)th eigenvalue \( F(n, x) \) is the probability distribution of the scaled position of the \( n \)th particle. Thus the functions \( F(n, x) \) provide us with the order statistics of the rightmost particles in the front with \( F(1, x) \) being the Tracy-Widom distribution [24]. Furthermore, the sum of the probability distributions \( F(n, x) \) gives us the scaled density \( \rho(x) \) of particles in the front. Figure 2 shows the distributions of the first six particles obtained by a powerful numerical method for the evaluation of Fredholm determinants [26]. The sum of them, shown by the dashed line, deviates from \( \rho(x) \) only at \( x \approx -8 \).

It is remarkable that the description of the front we found is consistent with a classical particle picture. One should also remember, however, that the position of the particles spreads out as \( t^{1/3} \) in the original (unscaled) variables. This subdiffusive spreading has a quantum origin; namely, it follows from the cubic nonlinearity of the dispersion around the Fermi points.

Additional information on the front region is contained in the cumulants \( \kappa_n \) of the FCS. They can be obtained from logarithmic derivatives \( \kappa_n = (-i \delta / \delta \lambda)^n \ln \chi(\lambda, s) |_{\lambda=0} \) of the generating function, Eq. (12). Using properties of Fredholm determinants, the cumulant generating function reads

\[ \ln \chi(\lambda, s) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\text{Tr}K^k}{k} (e^{i\lambda} - 1)^k, \]  

where the trace is defined as

\[ \text{Tr}K^k = \int_s^{\infty} dx_1 \ldots dx_k K(x_1, x_2) \ldots K(x_k, x_1). \]  

The first two cumulants, corresponding to the total number and to the fluctuations of the particle number, have simple forms \( \kappa_1 = \text{Tr}K \) and \( \kappa_2 = \text{Tr}K(1 - K) \), respectively. In general, carrying out the traces in \( \kappa_n \) is difficult and would require knowledge of the eigenvalues of the kernel. Having the spectrum would also give access to more complicated quantities of interest such as the entanglement entropy [10–12]

\[ S = -\text{Tr}[K \ln K + (1 - K) \ln(1 - K)]. \]  

Even though there exists a differential operator commuting with \( K \) [24], the solution of its eigenproblem is not known and the analytic calculation of \( \kappa_2 \) and \( S \) remains a challenge.

Numerically, we can calculate \( \kappa_2 \) and \( S \) from the matrix representation of the FCS given in Eq. (6), using matrices.
It would be important to check the universality of the edge behavior of quantum fronts. The simplest generalization would be to start from an initial state, where the left (right)-hand side of the chain is not completely filled (empty) [17] or the initial density profile is a smooth function [21]. In such cases the staircase structure has been found to be essentially unchanged. Moreover, a very similar structure of the magnetization front was observed in the transverse Ising model starting from a domain wall initial state [30]. One might expect that the FCS remains unchanged in the above cases.

Another important question is the role of interactions. Does the edge structure survive if one remains in the integrable Luttinger liquid regime? How robust is it against integrability breaking terms? Recently, a numerical technique was proposed [31] which might be suitable to attack these questions.

Finally, we mention an intriguing connection to the asymmetric exclusion process [32–34]. There, starting from a step initial condition, the distribution of particle positions can also be calculated [35]. One observes a $\sqrt{t}$ scaling for the width of the distributions at the edge of the front, while the $t^{1/3}$ scaling and the Tracy–Widom distributions emerge towards the bulk where the exclusion interaction between the particles becomes more important. Although there is no one-to-one correspondence to our result, nevertheless one may ask whether the quantum effects due to the curvature in the spectrum could be described in terms of some generalized semiclassical picture, e.g., by introducing effective interactions between particles of different velocities.

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