Nonequilibrium Phase Transitions

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Abstract

Nonequilibrium phase transitions are discussed with emphasis on general features such as the role of detailed balance violation in generating effective (long-range) interactions, the importance of dynamical anisotropies, the connection between various mechanisms generating power-law correlations, and the emergence of universal distribution functions for macroscopic quantities. Quantum spin chains are also discussed in order to demonstrate how to construct steady-states carrying fluxes in quantum systems, and to explain how the fluxes may generate power-law correlations.

CONTENTS

I. INTRODUCTION

There are many reasons for studying nonequilibrium phase transitions. Let us start be mentioning a few which carry some generality.

First and foremost, equilibrium in nature is more of an exception than the rule, and structural changes (which constitute a significant portion of interesting phenomena) usually take place in nonequilibrium conditions. Thus there is much to be learned about the complex ordering phenomena occurring far from equilibrium.

Second, while very little is understood about the general aspects of nonequilibrium systems, the equilibrium critical phenomena have been much studied and have been shown to display universal features. This universality emerges from large-scale fluctuations in such a robust way that one can expect that similar mechanisms will work in nonequilibrium situations as well. Thus, investigating the similarities and differences of equilibrium and nonequilibrium orderings may help to discover the distinguishing but still robust properties of nonequilibrium systems.

Third, power law correlations are present in many nonequilibrium phenomena and there have been many attempts to explain these correlations through general mechanisms. Closer examination, however, usually reveals a close connection to equilibrium or nonequilibrium critical phenomena.

My lectures were designed to revolve around problems related to the above points. The lectures are built on the theory of equilibrium phase transitions [1, 2], thus I assume knowledge about both static and dynamic critical phenomena at least on the level of familiarity with the basic concepts (symmetry breaking, order parameter, diverging correlation length, scaling, universality classes, critical slowing down, dynamical symmetries). I discuss simple examples throughout so that enterprising students could try out their luck implementing their ideas in simple calculations.

The lectures were delivered in Power Point presentation and the picturesque parts of the explanations are important for both understanding and memorizing. Thus I strongly encourage the reader to go through the slides of the lectures first and to turn to these notes as a last resort. The pdf files of the Power Point slides used in Les Houches can be found at http://dpm.univ-lyon1.fr/houches_ete/lectures/ or at http://poe.elte.hu/~racz/. In Lausanne, I expanded the material with details and cal-

culations and, furthermore, a closing lecture on pattern formation was included. The last lecture was mainly for visual fun and for the demostration of a few basic ideas. I did not develop notes for this part. The pdf files of the expanded lectures can be downloaded from the following addresses: http://theory.physics.unige.ch/homepage/enseignement.html and http://cgl.elte.hu/~racz/.

A. Nonequilibrium steady states

A general feature that distinguishes a nonequilibrium steady state (NESS) from an equilibrium one is the presence of fluxes of physical quantities such as energy, mass, charge, etc. Thus the study of NESS is, in a sense, a study of the effects of fluxes imposed on the system either by boundary conditions, or by bulk driving fields, or by some combination of them. A nonequilibrium steady stateswell known example is shown on Fig.1.
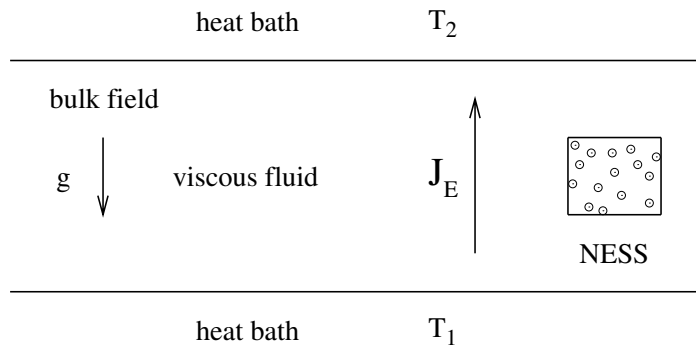
FIG. 1: Setup for Rayleigh-Bénard experiments.

This is the Rayleigh-Bénard experiment [3] in which a horizontal layer of viscous fluid (the system) is heated from below i.e. it has two heat baths of temperatures T_1 and T_2 attached (boundary conditions generating an energy flux). The presence of gravity (the bulk drive) is also important (it generates mass and momentum fluxes at large $\delta T = T_1 - T_2 > 0$).

For $T_1 = T_2$ this system relaxes into a quiescent equilibrium state while a small δT will also result in a quiescent state but it is already a NESS since energy flux is flowing through the system. Increasing δT, this steady state displays a nonequilibrium phase transition (Rayleigh-Bénard instability), first to a stationary convective pattern, and then to a series of more complicated structures which have fascinated researchers for the past century [3].

Starting from the Navier-Stokes equations, one can arrive at a mean-field level of under-

standing of the above phenomena. It is, however, not the level of sophistication one got used to in connection with equilibrium phase transitions. There, we have simple exactly solvable models such as e.g. the Ising model which give much insight into the mechanism of ordering and, furthermore, this insight can be used to develop theories which reveal the universal features of equilibrium orderings [1, 2].

The trouble with the Rayleigh-Bénard system is that we do not have a theory even for the NESS. The reason for this is that the fluxes result in steady state distributions , P_n^*, which break the detailed balance condition $w_{n \to n'} P_n^* = w_{n' \to n} P_{n'}^*$, where n and n' are two "microstates" and $w_{n \to n'}$ is the rate of the $n \to n'$ transition. As a consequence of the breaking of detailed balance, a NESS is characterized not only by the probability distribution, P_n^*, but also by the probability currents in the phase space. Unfortunately, we have not learned yet how to handle the presence of such loops of probability currents.

The main lesson we should learn from the Rayleigh-Bénard example is that, in order to have Ising type models for describing phase transitions in NESS, one should use models which relax to steady states with fluxes present. Such models have been developed during the last 20 years, and most of my lectures are about these stochastic models defined through "microscopic" elementary processes. The first level of description is in terms of master equations which are conceptually simple and allow one to make use of general results (uniqueness of stationary state, etc.) which in turn are helpful in defining dynamics that leads to NESS (Sec.II). The next level is to describe the same problem in terms of Langevin equations and develop field-theoretic techniques for the solution. Our discussions will include both levels of description and I hope that at the end an understanding will emerge about a few results which grew in importance in the last decade (generation of long-range interactions and effects of dynamical anisotropies (Sec.II), connection between mechanisms generating power-law correlations (Sec.III), and universality of distribution functions for macroscopic quantities (Sec.IV)).

Before starting, however, I would like to insert here a little essay about effective temperatures. This concept is being widely discussed in connection with slowly relaxing systems, the topic of this school. So it may be of interest to present here a view from the perspective of NESS.

B. Problems with usual thermodynamic concepts

Systems close to equilibrium may retain many properties of an equilibrium state with the slight complication that the intensive thermodynamic variables (temperature, chemical potential, etc.) become inhomogeneous on long lengthscales and they may slowly vary in time. This type of situations are successfully dealt with using the so called local equilibrium approximation [4], with the name giving away the essence of the approximation. The applicability of the concept of local equilibrium should diminish, however, as a system is driven far from equilibrium. Nevertheless, questions of "how large drive produces a far-enough state" and "couldn't one try to find a new equilibrium state near-by" are regularly asked and have legitimacy. So I will try to illuminate the problems on the example of the fluctuation-dissipation theorem much discussed nowadays due to attempts of associating effective temperatures with the various stages of relaxation in glasses [5] or with steady states in granular materials [6].

Let us consider a simple system of Ising variables σ with Hamiltonian $\mathcal{H}_0$ and in equilibrium at temperature $\beta = 1/(k_B T)$. Assuming that there is an external field H coupling linearly to the macroscopic magnetization $M = \sum_i \sigma_i$, one can write the equilibrium distribution function as

$$P_{eq}(\sigma) = Z^{-1} e^{-\beta \mathcal{H}_0(\sigma) + \beta H M(\sigma)} \tag{1}$$

The average value of the magnetization is given by

$$\langle M \rangle = Z^{-1} \sum_\sigma M(\sigma) e^{-\beta \mathcal{H}_0(\sigma) + \beta H M(\sigma)} \ . \tag{2}$$

and the static limit of the fluctuation-dissipation theorem is obtained as

$$\chi_M = \left. \frac{\partial \langle M \rangle}{\partial H} \right|_{H \to 0} = \beta \langle M^2 \rangle \tag{3}$$

where we assumed the system to be in the high-temperature phase ($\langle M \rangle = 0$). Note the simplicity and the accompanying generality of this derivation. It uses only the fact that the external field is linearly coupled to the quantity (M) we are considering.

The fluctuation-dissipation theorem is used in many ways. It helps simplify field-theoretic studies of fluctuations through diagrammatic expansions and it also gives a powerful checking procedure in both experiments and Monte Carlo simulations. Note that eq.(3) can also be

used to define the temperature of the system through $\beta = \chi_M / \langle M^2 \rangle$, and the temperature defined in this way would be the same when using different "M"-s and conjugate fields "H".

It is clear that a fluctuation-dissipation theorem generalized to NESS would be extremely useful. Let us now try to imagine how a similar relationship may arise when we drive the above system away from equilibrium (e.g. by attaching two heat baths of different temperatures). If the system relaxes to a NESS then there will be steady-state distribution function, (P) but the effective Hamiltonian $(\ln P)$ will contain all the interactions allowed by the symmetries of the system. Thus, assuming that the effective Hamiltonian can be expanded in H, one finds in the $H \to 0$ limit

$$P_{ne}(\sigma) \sim e^{-a\mathcal{H}_1(\sigma) + bH[M(\sigma) + S_3(\sigma) + ...]} \tag{4}$$

where S_3 is a notation for sums over all three-spin clusters with different couplings for different types of spatial arrangements of the three spins. Furthermore, a, b and all other newly generated couplings depend on the original couplings in $\mathcal{H}_0$ and on the temperatures of the heat baths.

Now a derivation of the fluctuation-dissipation theorem similar to the equilibrium case yields a more complicated equation

$$\chi_M = \left. \frac{\partial \langle M \rangle}{\partial H} \right|_{H \to 0} = a \left[\langle M^2 \rangle + \langle MS_3 \rangle + ... \right] . \tag{5}$$

There are two ways a simple form for the fluctuation-dissipation theorem may emerge from eq.(5). One is that a nonlinear field

$$Q = M + S_3 + ... \tag{6}$$

that is conjugate to H can be introduced (and effectively worked with). Then one obtains

$$\chi_Q = a \langle Q^2 \rangle \tag{7}$$

and thus a becomes the nonequilibrium β.

The other possibility is that a mean-field type decoupling scheme works well and then

$$\langle MS_3 \rangle = \langle M^2 \rangle f(C_2) \tag{8}$$

where $f(C_2)$ is a functional of the two-point correlations (and similar expressions are obtained for $\langle MS_{2n+1} \rangle$). Then equation (5) becomes

$$\chi_M = a\mathcal{F}(C_2)\langle M^2 \rangle . \tag{9}$$

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If the theory provides $\mathcal{F}(C_2)$ then the effective temperature can again be read of from the above generalized fluctuation-dissipation relationship.

There are problems with both lines of reasoning. Apart from the practical difficulties of nonlinear fields and the validity of mean-field type approaches, the main conceptual difficulty is the fact that changing from the magnetization to other fields (e.g. energy) the nonequilibrium version of the fluctuation-dissipation theorem leads to different values for the same "temperature" [10]. There are cases where the above schemes generalized to time-dependent processes work both at the theoretical [5, 7, 8] and the experimental levels [9] but there are clear examples when the concept of effective temperature does not apply [10]. Thus the meaning and use of nonequilibrium temperature has not been clarified enough to make a verdict on it.

II. PHASE TRANSITIONS FAR FROM EQUILIBRIUM

As mentioned in the Introduction, nonequilibrium phase changes constitute a large part of interesting natural phenomena and they are studied without worries about wider contexts. From a general perspective, on the other hand, the investigations of nonequilibrium phase transitions [11, 12] can be viewed as an attempt to understand the robust features of NESS. This view is based on the expectation that the universality displayed in critical phase transitions carries over to criticality in NESS as well. If this is true then studies of the similarities to and differences from equilibrium will lead to a better understanding of the role and general consequences of the dynamics generating NESS.

In the following subsections, we shall construct, describe, and discuss models which display nonequilibrium phase transitions. Apart from getting familiar with a few interesting phenomena, the main general conclusion of these discussions should be that dynamical anisotropies often yield dipole-like effective interactions [13–15] and, furthermore, competing non-local dynamics (anomalous diffusion) generates long-range, power-law effective interactions [16]. Along the way, we shall also understand that the detailed-balance violating aspects of local relaxational dynamics do not affect the universality class of the nonequilibrium phase transitions [15, 17].

A. Differences from equilibrium - constructing models with NESS

The violation of detailed balance has the consequence that not only the interactions determine the properties of the NESS but the dynamics also plays an important role. In order to understand and characterize the role of dynamics, a series of simple examples will be discussed in the following subsections.

First, let us discuss how to construct a model which yields a NESS in the long-time limit. A simple way is to attach two heat baths to a system, each generating a detailed-balance dynamics but at different temperatures. To see an actual implementation, let us consider how this is done for the one dimensional kinetic Ising model. This type of models have been much studied and a collection of mini-reviews about them can be found in [18].

The state of the system $\{\sigma\} \equiv \{\ldots, \sigma_i, \sigma_{i+1}, \ldots\}$ is specified by stochastic Ising variables $\sigma_i(t) = \pm 1$ assigned to lattice sites $i = 1, 2, \ldots, N$. The interaction is short ranged (nearest

neighbor) $-J\sigma_i\sigma_{i+1}$ and periodic boundary conditions ($\sigma_{N+1} = \sigma_1$) are usually assumed. The dynamics of the system is generated by two heat baths (labeled by $\alpha = 1, 2$) at temperatures T_α, meaning that the heat baths try to bring the system to equilibrium at temperature T_α by e.g. spin flips and spin exchanges, respectively.

Let us denote the rate of the flip of i-th spin ($\sigma_i \to -\sigma_i$) by $w_i^{(1)}(\{\sigma\})$, and let the rate of the exchanges of spins at sites i and j ($\sigma_i \leftrightarrow \sigma_j$) be $w_{ij}^{(2)}(\{\sigma\})$. Then the dynamics is defined by the following master equation for the probability distribution $P(\{\sigma\}, t)$:

$$\partial_t P(\{\sigma\}, t) = \sum_i \left[w_i^{(1)}(\{\sigma\}_i) \, P(\{\sigma\}_i, t) - w_i^{(1)}(\{\sigma\}) \, P(\{\sigma\}, t) \right] +$$
$$\sum_{ij} \left[w_{ij}^{(2)}(\{\sigma\}_{ij}) \, P(\{\sigma\}_{ij}, t) - w_{ij}^{(2)}(\{\sigma\}) \, P(\{\sigma\}, t) \right] \tag{10}$$

where the states $\{\sigma\}_i$ and $\{\sigma\}_{ij}$ differ from $\{\sigma\}$ by the flip of the i-th spin and by the exchange of the i-th and j-th spins, respectively.

The assumption that the dynamics is generated by heath baths means that the rates satisfy detailed balance at the appropriate temperatures:

$$w_{i(j)}^{(\alpha)}(\{\sigma\}) \, P_\alpha^{eq}(\{\sigma\}) = w_{i(j)}^{(\alpha)}(\{\sigma\}_{i(j)}) \, P_\alpha^{eq}(\{\sigma\}_{i(j)}) \quad , \tag{11}$$

where $P_\alpha^{eq} \sim \exp\left[-J/T_\alpha \sum_i \sigma_i \sigma_{i+1}\right]$ is the equilibrium distribution of the Ising model at temperature T_α. Eq.(11) leaves some freedom in the choice of w^α-s, and one is usually guided by simplicity. The most general spin flip rate that depends only on neighboring spins has the following form [19]

$$w_i^{(1)}(\sigma) = \frac{1}{2\tau_1} \left[1 - \frac{\gamma}{2}\sigma_i \left(\sigma_{i+1} + \sigma_{i-1}\right)\right] \left(1 + \delta\sigma_{i+1}\sigma_{i-1}\right) \quad . \tag{12}$$

Without any other heath baths, equations (10) and (12) define the Glauber model [19] which relaxes to the equilibrium state of the Ising model at temperature T_1 defined through $\gamma = \tanh(2J/k_B T_1)$. The time-scale for flips is set by τ_1 and δ is restricted to the interval $-1 < \delta < 1$.

The competing dynamical process is the generation of spin exchanges (Kawasaki dynamics [20]) by a second heath bath at a temperature $T_2 \neq T_1$. In the simplest case, the exchanges are between nearest neighbor sites and the rate of exchange satisfying detailed balance (11) is given by

$$w_i^{(2)}(\sigma) = \frac{1}{2\tau_2} \left[1 - \frac{\gamma_2}{2} \left(\sigma_{i-1}\sigma_i + \sigma_{i+1}\sigma_{i+2}\right)\right] \quad . \tag{13}$$

where $\gamma_2 = \tanh(2J/k_BT_2)$ and τ_2 sets the timescale of the exchanges. It is often assumed that the exchanges are random ($T_2 = \infty$) and thus $w_i^{(2)}(\sigma) = 1/(2\tau_2)$.

Equations (10), (12) and (13) define a model which can be shown to have a NESS and one can start to ask questions about the phase transitions in this steady state. The generalizations to higher dimensions, to various combinations competing dynamics (flip - flip, flip - exchange, exchange - exchange), and other types of dynamical steps (resulting e.g. from a bulk driving field) should be obvious. Just as it should be obvious that there are not too many exactly solvable models in this field and most of the results are coming from simulations [11, 12].

Before turning to results, let us also introduce a Langevin equation description of competing dynamics. The Langevin approach has been successful in dynamic critical phenomena [2] where the counterparts of the Glauber and Kawasaki models are called Model A and B, correspondingly. This correspondence makes the "two-heath-baths" generalization straight-forward. The coarse grained magnetization of the Ising model is replaced by n-component order-parameter field, $S^i(\mathbf{x}, t)$, ($i = 1, ..., n$) ($n = 1$ is the Ising model, and $n \to \infty$ is the spherical limit that allows simple analytic calculations as shown below). The system evolves under the combined action of local relaxation (Model A) satisfying detailed balance at temperature T_1 and diffusive dynamics (Model B) at temperature T_2 which yields the following Langevin equation for the Fourier transform $S_{\mathbf{q}}^i(t)$

$$\partial_t S_{\mathbf{q}}^i = -\mathcal{L}_{\mathbf{q}}^{(1)} S_{\mathbf{q}}^i + \eta_1^i(\mathbf{q}, t) - \mathcal{L}_{\mathbf{q}}^{(2)} S_{\mathbf{q}}^i + \eta_2^i(\mathbf{q}, t) \quad . \tag{14}$$

Here $\mathcal{L}_{\mathbf{q}}^{(\alpha)} S_{\mathbf{q}}^i = \Gamma_{\mathbf{q}}^{(\alpha)} \delta F^{(\alpha)} / \delta S_{-\mathbf{q}}^i$ with $F^{(\alpha)}$ being the free energy at temperature T_α, and the kinetic coefficient $\Gamma_{\mathbf{q}}^{(\alpha)}$ is enforcing the conservation laws. In particular, $\Gamma_{\mathbf{q}}^{(\alpha)} = \Gamma_0^{(\alpha)}$ in Model A without conservation laws, and $\Gamma_{\mathbf{q}}^{(\alpha)} = D^{(\alpha)} \mathbf{q}^2$ for Model B with diffusive dynamics conserving the total magnetization. In case $F^{(\alpha)}$ is the Landau-Ginsburg functional, we have

$$\mathcal{L}_{\mathbf{q}}^{(\alpha)} S_{\mathbf{q}}^i = \Gamma_{\mathbf{q}}^{(\alpha)} \left[(r_0^\alpha + q^2) S_{\mathbf{q}}^i + u \sum_{j=1}^n \int_{\mathbf{q}'} \int_{\mathbf{q}''} S_{\mathbf{q}'}^j S_{\mathbf{q}''}^j S_{\mathbf{q}-\mathbf{q}'-\mathbf{q}''}^i \right] \tag{15}$$

where r_0^α is linear in T_α and $u \sim 1/n$ in the $n \to \infty$ spherical limit. In order to ensure that in case of a single heat bath, the system relaxes to equilibrium satisfying detailed balance, the noise terms in eq.(14 are Gaussian-Markovian random forces with correlations of the form:

$$\langle \eta_\alpha^i(\mathbf{q}, t) \eta_{\alpha'}^j(\mathbf{q}', t') \rangle = 2\Gamma_{\mathbf{q}}^{(\alpha)} \delta_{\alpha\alpha'} \delta_{ij} \delta(\mathbf{q} + \mathbf{q}') \delta(t - t') \quad . \tag{16}$$

Eqs. (14),(15) and (16) define the model for the particular competing dynamics chosen and we are now ready to deduce some features of the NESS generated. Of course, just as in case of kinetic Ising models, the number of possible competing dynamics is infinite and the question is whether conclusions of some generality could be reached.

B.   Generation of long-range interactions - nonlocal dynamics

The remarkable consequences of competing dynamics can be seen already on the example of $d = 1$ flip-and-exchange model which may produce ordering even though the interactions are of short range. Indeed, if T_1 temperature spin flips are competing with $T_2 = \infty$ spin exchanges of randomly chosen pairs then the system orders below a certain T_{1c} [16].

It turns out that the transition is of mean-field type and this gives a clue to understanding. Indeed, let us imagine that the rate of spin exchanges is large compared to the rate of flips. Then the random exchanges mix the spins in between two flips and the flipping spin sees the "average spins" in its neighborhood - a condition for mean-field to apply.

The mean field result can also be interpreted as the generation of infinite-range effective interactions. This interpretation can be put on more solid base by studying the above model with $T_2 = \infty$ spin exchanges where the probability of exchange at a distance r is decaying with r as $p(r) \sim 1/r^{d+\sigma}$ (the spins exchanges are σ dimensional Levy flights in dimension d). The system orders again below a T_{1c} and the examination of the critical exponents reveals [16] that the transition is in the universality class of long-range interactions decaying with r as $J(r) \sim 1/r^{d+\sigma}$. It is important to note that the above results are nonequilibrium effects which would disappear if the spin exchanges would also be at T_1.

Let us now see if the same results can be derived from the Langevin equation approach. The spin flips are translated into the Model A part of the dynamics while the Levy flights can be represented [21] by anomalous diffusion with $\Gamma_{\mathbf{q}}^{(2)} = D^{(2)}\mathbf{q}^2$ replaced by $\Gamma_{\mathbf{q}}^{(2)} = D^{(2)}\mathbf{q}^\sigma$ with $0 < \sigma < 2$ being the dimension of the Levy flight. Thus the Langevin equation becomes

$$\dot{S}_{\mathbf{q}}^i = -\Gamma_0(r_0 + q^2)S_{\mathbf{q}}^i \ -\Gamma_0 u \sum_{j=1}^{n} \int_{\mathbf{q}'} \int_{\mathbf{q}''} S_{\mathbf{q}'}^j S_{\mathbf{q}''}^j S_{\mathbf{q}-\mathbf{q}'-\mathbf{q}''}^i + \eta_{\mathbf{q}}^i(t)$$

$$-Dq^\sigma S_{\mathbf{q}}^i + \bar{\eta}_{\mathbf{q}}^i(t) \ . \tag{17}$$

Note that due to the randomness of the Levy flights ($T_2 = \infty$), the interaction and the nonlinear terms are missing in the Levy flight part (second line) of the equation. As discussed

in Sec.II A, the η-s are Gaussian-Markoffian random forces with correlations $\langle \eta_{\mathbf{q}}^i(t)\eta_{\mathbf{q}'}^j(t')\rangle = 2\Gamma_0\delta_{ij}\delta(\mathbf{q}+\mathbf{q}')\delta(t-t')$ and $\langle \bar{\eta}_{\mathbf{q}}^i(t)\bar{\eta}_{\mathbf{q}'}^j(t')\rangle = 2Dq^\sigma\delta_{ij}\delta(\mathbf{q}+\mathbf{q}')\delta(t-t')$.

In order to see the generation of long range interaction in the above model, let us first make an exact calculation [16] in the spherical limit ($n \to \infty$) where fluctuations in $u\sum S_{\mathbf{q}}^j(t)S_{\mathbf{q}'}^j(t)$ may be neglected and this quantity may be replaced by

$$u\sum_{j=1}^{n}\langle S_{\mathbf{q}}^j(t)S_{\mathbf{q}'}^j(t)\rangle = unC(q,t)\delta(\mathbf{q}+\mathbf{q}') \quad , \tag{18}$$

where the brackets $\langle\rangle$ denote averaging over both the initial conditions and the noises η and $\bar{\eta}$ (we restrict ourselves to the study of the high-temperature phase where the dynamic structure factor $C(q,t) = \langle S_{\mathbf{q}}^j(t)S_{-\mathbf{q}}^j(t)\rangle$ is independent of j).

The decoupling (18) leads to a linear equation of motion and so the self-consistency equation for $C(q,t)$ can be easily derived

$$C(q,t) = 2(\Gamma_0 + Dq^\sigma)\int_0^t dt' e^{-2\int_0^{t'}\left\{\Gamma_0[r_0+q^2+unC(q,s)]+Dq^\sigma\right\}ds} . \tag{19}$$

Here, the initial condition $C(q,0) = 0$ was used for simplicity. The $t \to \infty$ limit does not depend on the initial condition and the equation for the steady state structure factor $C(q) = C(q, t \to \infty)$ becomes

$$C(q) = \frac{\Gamma_0 + Dq^\sigma}{\Gamma_0(r_0 + q^2 + unS) + Dq^\sigma} \quad , \tag{20}$$

where $S = \int d\mathbf{q}C(q)$.

The long-wavelength instabilities are determined by the $q \to 0$ form of $C(q)$ which for $0 < \sigma < 2$ can be written as

$$C(q) \approx (r_0 + \lambda q^\sigma + unS)^{-1} \quad , \tag{7}$$

with $\lambda = D/\Gamma_0$. This form coincides with the long-wavelength limit of the equilibrium structure factor of a spherical model in which the interactions decay with distance as $r^{-d-\sigma}$. Consequently, both the self-consistency equation for $r = r_0 + unS$ and the critical behavior that follows from it are identical to that of the equilibrium long-range model. Thus we can conclude that the critical properties of the NESS are dominated by an effective long-range potential proportional to $r^{-d-\sigma}$.

The above conclusion should be valid quite generally for finite n as well. Looking at eq.(17), one can see that the correlations in the effective noise ($\eta_{eff} = \eta + \bar{\eta}$) have an

amplitude $2(\Gamma_0 + Dq^\sigma)$. One expects that the Dq^σ term can be neglected in the long-wavelength limit and thus that the noise $\bar{\eta}$ in the Levy-flight exchanges can be omitted. Without $\bar{\eta}$, however, the system described by equation (2) satisfies detailed balance and has an effective Hamiltonian which, apart from the usual short-range interaction pieces, contains the expected long-range part $\lambda \int d\mathbf{q} q^\sigma \sum_i S^i(\mathbf{q}) S^i(-\mathbf{q})$.

We should note here that $\sigma \to 2$ corresponds to usual diffusion and that the above arguments changes for $\sigma = 2$ since no long-range interactions are generated any more, and no change in critical behavior occurs. This result is another way of saying that Model A type dynamics is robust against diffusive perturbations which break detailed balance [17]. Note also that if both the competing dynamics are relaxational then, adding up the corresponding deterministic and noise parts in the Langevin equation (14), one can easily deduce that the breaking of the detailed balance does not change the universality class of the equilibrium phase transition.

C.   Generation of long-range interactions - dynamical anisotropies

In order to understand the meaning of dynamical anisotropy, let us consider the two-temperature, diffusive kinetic Ising model [22] on a square lattice. Two heat baths are attached and both of them generates nearest neighbor spin exchanges. Exchanges along one of the axis (called 'parallel' direction) satisfy detailed balance at temperature $T_\parallel$ while exchanges in the 'perpendicular' direction are produced by a heat bath of temperature $T_\perp$. It is important to note that the interactions $J\sigma_i\sigma_j$ are the same along both axes. It is the dynamics that is anisotropic.

For $T_\perp = T_\parallel = T$, this is the Kawasaki model [20] which relaxes to the equilibrium Ising model at T and, consequently, it displays a continuous transition. Since the dynamics conserves the total magnetization, the ordering for $T < T_c$ appears as a phase separation.

For $T_\perp \neq T_\parallel$, on the other hand, there is a flow of energy between the $\parallel$ and $\perp$ heath baths and the system relaxes to a NESS. MC simulations [14, 22] show that a critical phase transition is present for $T_\perp \neq T_\parallel$ as well but the phase separation is distinct from that occurring in equilibrium. The interfaces between the domains of up and down spins align with normals along the directions of lower temperatures. Thus the symmetries of the ordered states are different from the symmetry of the equilibrium order where interfaces with normals

along any of axes coexist (isotropic ordering). As a consequence, the universality classes of the $\parallel$ and $\perp$ orderings are found to be distinct from the Ising class [22, 23]. Renormalization group calculations actually show that the universality class of the nonequilibrium ordering is that of a uniaxial ferromagnet with dipolar interactions [24].

A dramatic demonstration of the long-range nature of the interactions generated by anisotropic dynamics comes from the generalization of the above model to $n = 2$ component spins ($2T - XY$ model). One finds that the NESS in this system displays an ordering transition [14], a fact that would be in contradiction with the Mermin-Wagner theorem [25] should the effective interaction be short-ranged.

Let us try now understand the generation of the dipole-like interactions using the Langevin equation approach and considering the spherical limit again. The two-temperature, diffusive Ising model corresponds to competition of two Model B type dynamics along $\parallel$ and $\perp$ axes. Thus the equation of motion is $\dot{S}_{\mathbf{q}}^i(t) = \mathcal{L}_\parallel(q)S_{\mathbf{q}}^i + \eta_\parallel^i(\mathbf{q}, t) + \mathcal{L}_\perp(q)S_{\mathbf{q}}^i + \eta_\perp^i(\mathbf{q}, t)$ with the diffusion in the $\alpha = \parallel, \perp$ directions described by the corresponding $\mathcal{L}_\alpha$ terms:

$$\mathcal{L}_\alpha S_{\mathbf{q}}^i = D_\alpha q_\alpha^2 \left[(r_0^\alpha + q^2)S_{\mathbf{q}}^i + u \sum_{j=1}^n \int_{\mathbf{q}'} \int_{\mathbf{q}''} S_{\mathbf{q}'}^j S_{\mathbf{q}''}^j S_{\mathbf{q}-\mathbf{q}'-\mathbf{q}''}^i \right] . \tag{21}$$

where $\mathbf{q} = (q_\parallel, q_\perp)$. Note the isotropy of the interaction (q^2 term) and the different temperatures (r_0^α) for diffusion in different directions. The noise correlations follow from detailed balance requirements, $\langle \eta_\alpha^i(\mathbf{q}, t)\eta_{\alpha'}^j(\mathbf{q}', t')\rangle$
$= 2D_\alpha q_\alpha^2 \delta_{\alpha\alpha'}\delta_{ij}\delta(\mathbf{q} + \mathbf{q}')\delta(t - t')$.

Just as in the kinetic Ising model for $r_0^\parallel = r_0^\perp$, we have Model B with anisotropic diffusion (not a dynamical anisotropy!) and with an equilibrium steady state, while a NESS is produced for $r_0^\parallel \neq r_0^\perp$. The nature of the phase transitions in the NESS becomes transparent in the spherical limit ($n \to \infty$ and $u \sim 1/n$) where the fluctuations in $u \sum S_{\mathbf{q}}^j(t)S_{\mathbf{q}'}^j(t)$ may be neglected. This linearizes the equation of motion and allows to write down a selfconsistency equation for $C(\mathbf{q}) = \langle S_{\mathbf{q}}^j(t)S_{\mathbf{q}'}^j(t)\rangle$. The $t \to \infty$ limit then yields the steady state structure factor in the following form [14]

$$C(\mathbf{q}) = \frac{q_\parallel^2 + aq_\perp^2}{q_\parallel^2(r_0^\parallel + q^2 + S) + aq_\perp^2(r_0^\perp + q^2 + S)} \quad , \tag{22}$$

where $a = D_\perp/D_\parallel$ and $S = un \int d\mathbf{q}\, C(\mathbf{q})$.

One can see now the origin of dipole-like effective interactions. Because of the dynamical anisotropy, the $\mathbf{q} \rightarrow 0$ limit of $C(\mathbf{q})$ is different whether first $q_\parallel \rightarrow 0$ and then $q_\perp \rightarrow 0$ or vice versa. This singularity of the long-wavelength limit translates into such power law correlations in real space which are characteristic of dipole interactions in equilibrium systems. Hence the conclusion [14, 24] that the dynamical anisotropy has generated dipole-like interactions. Note that this is a nonequilibrium effect. The long-wavelength singularity disappears as soon as the heat baths have equal temperatures ($r_0^\parallel = r_0^\perp$).

The dynamical anisotropy is a strong effect and its mechanism of action is rather simple as we have seen above. Accordingly, it is the most viable candidate to change the universality class of equilibrium phase transitions by breaking detailed balance [15].

D. Driven lattice gases, surface growth

Models of NESS have a long history but the first one that became the center of attention and was recognized as the "Ising model" of NESS was the driven lattice gas (see [11] and references therein). The model can be understood as a kinetic Ising model with the up-spins being the particles in the lattice gas. Spin exchange dynamics at temperature T represents the particle diffusion and an external bulk field E_x drives the up-spins (particles) along one of the lattice axes (x). In order to have a NESS with particle current one must also use periodic boundary conditions in the field direction.

This model displays a critical phase transition in its NESS and the phase separation that follows at low T is characterized by strong anisotropy: the interfaces align parallel with E_x. Thus one can see some similarities with the two-temperature diffusive model of Sec.II C.

One can easily recognize that dynamical anisotropy is at work here. The driving field can be considered as a second heath bath which generates the essential part of the dynamics along the x direction. There is a difference, however, from the two-temperature model of Sec.II C in that the drive now has a directionality (the forward-backward symmetry of diffusion is broken). As a result the phase transition is expected to belong to a new (nonequilibrium) universality class distinct from that of the dipole class. This is indeed what has been obtained in renormalization group calculations [11]. Unfortunately, the structure of the long-wavelength singularities in the two systems are similar and thus there are difficulties in observing the differences in numerical work. This problem has generated some debate that

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is still going on [26]. We believe the debate will not modify the general picture summarized in [11], and it will not change the conclusion about the importance of dynamical anisotropy.

An interesting and important field where even the simplest systems show "effective" critical behavior due to the unbounded long-wavelenth fluctuations is the field of surface growth processes. Most of the roughening transitions and transitions between various rough phases are genuine nonequilibrium phase transitions and have been much studied [27, 28]. Remarkably, however, these transitions have not provided new insight into the general features of nonequilibrium criticality, they merely confirmed that the dynamics and dynamical anisotropy play an important role in determining the universality classes of growth processes.

One of the unsolved problems that continues to fascinate researchers is the phase transition in the Kardar-Parisi-Zhang equation [29]. We shall discuss the problem of growing surfaces including KPZ equation in connection with the nonequilibrium distribution functions in Sec.IV.

E. Flocking behavior

Up to this point, we have considered usual physical systems driven out of equilibrium. Here I would like to give a taste of what awaits one if the studies are extended to the living realm.

Living creatures can also be viewed as units attached to two heat baths. One of them is the internal energy source which on a short time-scale is an infinite bath from which energy can be drawn at a given rate. The other one is the surroundings to which energy is lost by dissipation (friction, heat loss, ...) due to the activity of the unit. This view suggests that a collection of such self-propelled units will show orderings (nonequilibrium phase transitions) depending on the interactions between the units, on their density, on their possible motions and on the dissipation mechanisms. Indeed, collective behavior is often observed in flocks of birds, in schools of fish, in swimming cells, etc. and, as shown below, some of these phenomena can be described in terms of a surprisingly simple model.

Model [30] was introduced to describe the collective motion of self-propelled particles with birds and bacteria being candidates for these particles. We shall use the language of "birds" below.

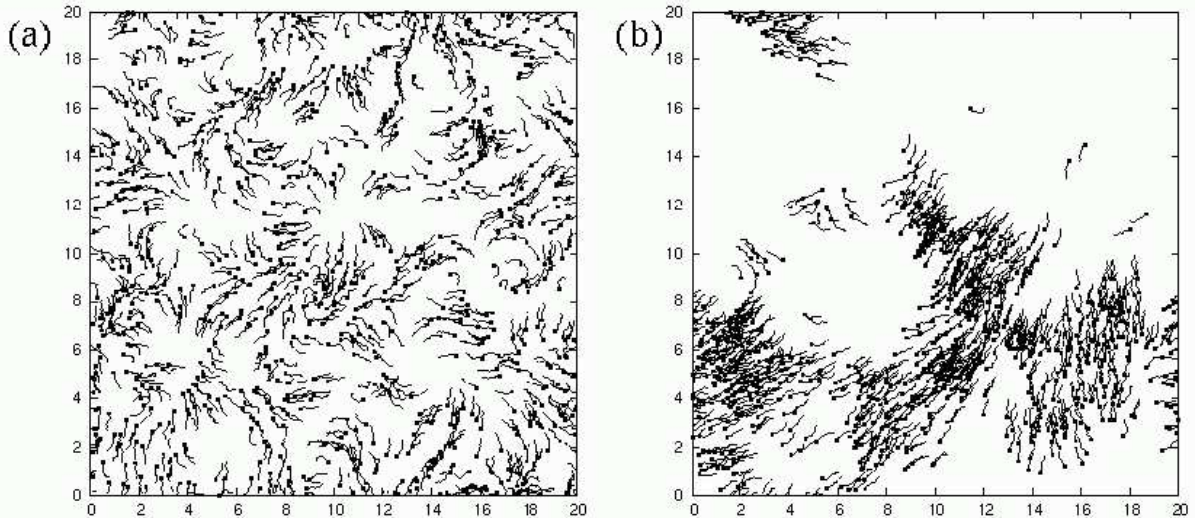The basic assumptions of the model are that (i) the birds fly with constant speed $|\vec{v}_i| = 1$

FIG. 2: Flocking: Trajectories of 10000 self-propelled particles in the model described in the text. The parameters are chosen so that the stationary order parameter is $\phi = 0.8$. Each particle is represented by a point marking its current position as well as a continuous line showing its recent (10 time step long) trajectory. (a) Initial stage of the relaxation, (b) the stationary regime. Pictures are curtesy of A. Czirók and T. Vicsek.

(i is the bird index), and (ii) the birds adjust their direction θ_i in time intervals of $\tau = 1$ to the average direction of other birds within a distance r

$$\theta_i(t + \tau) = \langle \theta(t) \rangle_r + \eta_i \tag{23}$$

where η_i is random noise with amplitude η.

Assumption (i) handles the energy in- and outflow by strictly equating them, while assumption (ii) handles the interactions by seemingly reducing them to interactions in the space of velocity directions. This is not quite so, however, since the motion of the birds $\vec{x}_i(t + \tau) = \vec{x}_i(t) + \vec{v}_i(t)\tau$ couples the directional and spatial motions.

The control parameters in the system are r, η and the density of particles ρ. Keeping r and ρ fixed while varying the "temperature" η, one finds that the birds are flying randomly for $\eta > \eta_c(r, \rho)$ while collective motion develops below η_c where the birds tend to move in the same direction. An order parameter characterizing this spontaneously symmetry breaking can be chosen e.g. $\phi = |\sum_i^N \vec{v}_i|/N$. Fig.2 shows the time evolution deep in the ordered

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regime ($\phi = 0.8$) starting from a random configuration. One can see that local orientational order develops in the initial stages of relaxation (the state here shows resemblance to the states in classical XY ferromagnet) while the stationary state with almost full orientational order shows large density fluctuations. The structure and the large density fluctuations observed in the ordered state and, furthermore, the measurements of the critical exponents of the transition [30] suggest that the ordering in this system is in a universality so far not encountered.

A remarkable field theory has also been constructed for flocking [31]. It is a generalized Navier-Stokes equation with additional Model A type terms which drive the velocity to $\langle |\vec{v}_i| \rangle = 1$. This theory explains the large density fluctuations present in the ordered state. The investigation of ordering transition is at a higher level of difficulty, however, and has not been completed yet.

Clearly, much remains to be done before we understand flocking and before the model can be compared with experiments quantitatively. Nevertheless, activity is expected in this direction since the model of flocking is not much more complicated than the more standard NESS models discussed above, and, at the same time, it has close connection with experimentally observable, truly "far-from-equilibrium" phenomena. Hopefully, by designing and understanding similar models, a kind of "universality map" of the collective dynamics of self-driven units can be found.

III. WHERE DO THE POWER-LAWS COME FROM?

Systems displaying power law behavior in their various characteristics (correlation in space or time, fluctuation power spectra, size-distributions, etc.) are abundant in nature. The most impressive examples are found in biology (e.g. the metabolic rate vs. mass relationship for living creatures displays scaling over 28 decades [32]) but there are remarkable examples in solid state physics (power spectra of voltage fluctuations when a current is flowing through a resistor [33] - 6 decades of scaling), in geology (the number of earthquakes vs. their magnitude [34] - 5 decades) and scaling over 2-3 decades is seen everywhere [see e.g. the white-dwarf light emission [35], the flow of sand through an hourglass [36], the number of daily trades in the stock market [37], water flow fluctuations of rivers [38], the spike trains of nerve cells [39], the traffic flow on a highway [40], interface fluctuations [27], dissipation in the turbulent systems [41]].

Understanding the (possibly) common origins of scaling in the above phenomena appears to be a highly nontrivial task. Power laws, of course, arise naturally in critical phenomena and we understand them: their origins are in the diverging fluctuations at the critical point. Thus the first question one may ask is the following.

- Can the power laws just be the result of nonequilibrium phase transitions and the associated critical behavior?

In equilibrium systems, however, one must tune a parameter to its critical value in order to observe scale-invariant behavior while nonequilibrium systems appear to be in scale-invariant states without any tuning. Thus the answer to the first question appears to be negative.

The wide variety of the phenomena in the above list suggests that the next question could be as follows.

- Can the scaling merely be a natural outcome of complex dynamics?

After all, we have seen in Sec.II C that competing dynamics may generate long-range (power-law) interactions which may be at the origin of scaling even away from a critical point. The answer to this question may be a yes but, unfortunately, many of the problems mentioned are not amenable to an analysis in terms of simple competing dynamics and then the following question remains unanswered:

- What are the ingredients of complex dynamics which determine the existence and the characteristics (e.g. exponents) of the power laws?

There is an attempt to answer all the above question in affirmative along a logic that begins with the notion of self-organized criticality (SOC) introduced by Bak, Tang and Wiesenfeld [42]. According to this notion, systems with complex dynamics tune themselves to a state with a kind of avalanche type dynamics that is underlying a large number of scale-invariant phenomena. The notion of SOC has now been understood in terms of an interplay between local and non-local dynamics [43] which indeed tunes the system [44] to a nonequilibrium (absorbing-state) critical point. Then the problem of SOC is reduced to investigating the absorbing state transitions [45] and the characteristics of power laws can be determined by studying the universality of absorbing state transition. This is an interesting and active field of research and it is worth understanding the main points. Accordingly, I will discuss SOC in the next subsection, and will explain the connection to absorbing state transitions in the following one.

A. Self-organized criticality (SOC)

The first model of SOC was introduced to describe sandpiles. Later developments, however, made energy packets from the grains of sand, so the balls in Fig.3 will be grains at the beginning but will be called energy packets later. The dynamics defining the model consists of local and non-local elements. The sites of a (usually) two-dimensional lattice are occupied by grains and the local aspect of their dynamics is in the redistribution of the grains. If a site contains more than z_c grains (e.g. $z_c = 4$ on a square lattice) then the site is active and z_c grains are redistributed to the neighboring lattice sites. The redistribution of particles (avalanche) continues until active sites are found. Clearly, an avalanche stops after a while since the redistribution leads to loss of particles at the boundaries (or, in terms of the energy model, dissipation occurs at the boundaries). Once the avalanche stopped, an external supervisor notices it (this is the nonlocal part of the dynamics) and starts to add new particles (energy is injected into the system) until a new avalanche starts.

The above dynamics yields a stationary state in the long-time limit, and the steady state characteristics of avalanches can be measured. Such a characteristics is e.g. the number of sites s which become active during the process, and the remarkable feature of this model is
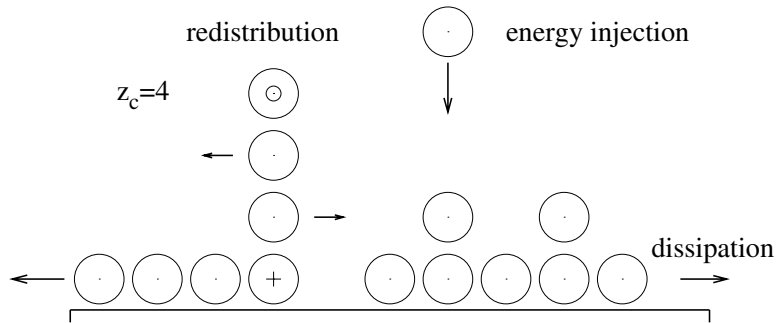
FIG. 3: Sandpile model. Particles (energy packets) are deposited on a two-dimensional substrate (one-dimensional section is shown). Injection stops when a site becomes active, i.e. it is occupied by more than z_c particles. Then redistribution to neighboring sites take place and particles disappear (dissipation of energy) at the edges. The process continues until all active sites are eliminated. Then the particles source is switched on again.

that the distribution of s (and of other quantities such as the spatial size and the lifetime of the avalanches) is found to display a power law form

$$P(s) \sim s^{-\tau} . \tag{24}$$

Thus one discovers that although the model does not contain parameters to tune, nevertheless it shows critical behavior (z_c can be changed without changing the criticality of the outcome). This observation generated a large amount of activity and *criticality-without-tuning* was seen in a number of similar models [46]. The resulting notion of *self-organized criticality* grew in importance [47] and, accordingly, new effort was put into understanding how SOC works.

An important feature that was recognized quite early [43] is the existence of a *non-local supervisor* who watches the activity of the avalanches and, upon ceasing of the activity, switches on the source of particles (or of energy). In principle, non-local dynamics can generate long-range correlations in both time and space so the emergence of criticality is not necessary a surprise. Viewing the problem from another angle, the non-local dynamics separates the timescales of the avalanches and of the particle injection. Thus, in practice, there is tuning. Namely, the system is considered in the limit of particle injection rate going to zero (actually, the dissipation is also tuned to zero since the particles disappear only at the boundaries of the system).

The zero injection rate, however, does not have to be a critical point, and the next important development was [44] the demonstration that it is indeed a nonequilibrium (absorbing state) critical point.

B. Absorbing state transitions and their connection to SOC

Absorbing state transitions appear in many contexts in nonequilibrium statistical physics [12, 48], and they are studied intensively since they are thought to be one of the truly nonequilibrium phenomena without counterpart in equilibrium systems. In order to understand the basics of it, let us consider a fixed energy sandpile model [49] shown on Fig.4.
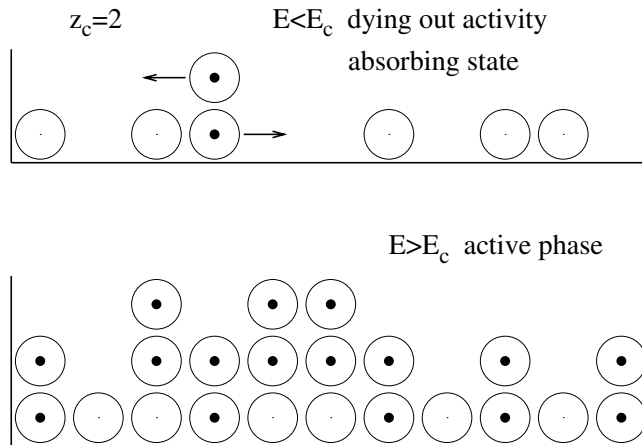

FIG. 4: A one-dimensional fixed energy sandpile model. Dynamics is defined by the energy (particles) being redistributed if a site contains more than z_c units of energy (the units which are redistributed in the next time-step are marked by large dots in their center). The total energy of the system is conserved since the boundaries prevent the loss of energy.

This model differs from the sandpile model by calling the particles energy units, and by the absence of both the injection and the dissipation of energy (particles). Thus the total energy E is conserved and as one can easily see from Fig.4 the behavior of the system is essentially different at small and large values of E. At small $E < E_c$, the activity (redistribution) ceases in the long-time limit and the system falls into a so called *absorbing state*. For large $E > E_c$, on the other hand, there are always active sites and the redistribution continues forever. For $t \rightarrow \infty$, the system settles into a steady state which is called *active state* and

the *activity* can be quantified by measuring e.g. the number of active sites. One finds then that the absorbing state transition (i.e. the absorbing-active state transition) is a critical phase transition with the *activity* changing continuously through the transition point E_c (see Fig.5).
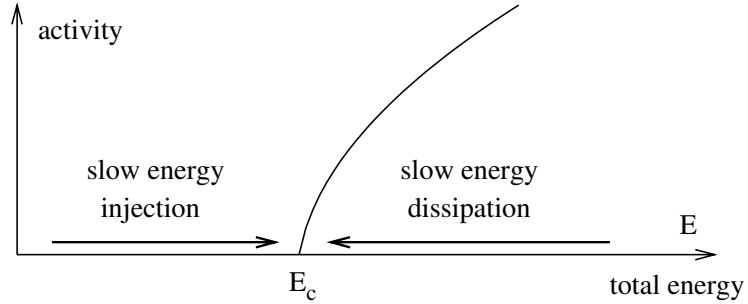

FIG. 5: Activity as a function of the total energy for the fixed energy sandpile model described on Fig.4. The evolution of the system as dissipation at the boundaries or the energy injection is switched given by the left and right arrows, respectively.

Once the absorbing state transition is understood, it is easy to make the connection to SOC. Indeed, let us assume that we have the fixed energy sandpile in the active state $(E > E_c)$ and let us switch on the dissipation at the boundaries. Then the energy decreases slowly (note that the dissipation is proportional to the surface of the sample while E is proportional to the volume). This lowering of energy will continue until E reaches just below E_c when the system falls into the absorbing state and thus the dissipation stops.
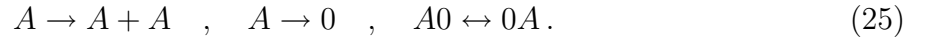
Let us now return to the fixed energy sandpile but this time let us start from an absorbing state $(E < E_c)$ and switch on the "external supervisor" who is injecting energy into the system. The supervisor is required to stop the injection if adding of the last energy packet started activity in the system. This process increases the energy E infinitesimally slowly and brings the system near and perhaps slightly past the threshold of activity $E = E_c$.

Now, if both the dissipation at the boundaries and the "external supervisor" are present then the fixed energy sandpile model is nothing else but the sandpile model generating SOC. And we see that SOC emerges because the combined action of the dissipation and the "supervisor" brings the system to the critical point of the absorbing state transition of the fixed energy sandpile model.
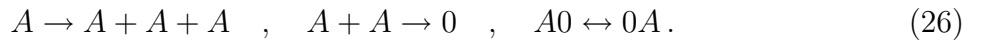
The mechanism unmasked above is rather general and present in many models of self

organized criticality [44, 50]. The value of recognizing this mechanism lies in making it possible to describe and calculate scaling properties of SOC by studying "usual" nonequilibrium phase transitions. In particular, one may hope that field-theoretic description of SOC may be obtained through studies of the appropriate absorbing state transitions.
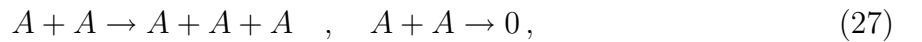
Of course, absorbing state transitions are numerous and it is not obvious which one is in the same universality class as a given system displaying SOC. In general, continuous phase transitions to an absorbing state are in the universality class of *directed percolation* [51–53] that can be described by the following reaction diffusion process

$$A \to A + A \quad , \quad A \to 0 \quad , \quad A0 \leftrightarrow 0A \,. \tag{25}$$

Directed percolation is rather robust to various changes in its rules but the presence of extra symmetries (conservation laws) may change the universality class of an absorbing state transition. A well known example is the *parity conserving process* [54–56] which has the following reaction-diffusion representation

$$A \to A + A + A \quad , \quad A + A \to 0 \quad , \quad A0 \leftrightarrow 0A \,. \tag{26}$$

Both of the above processes have been much investigated and the scaling properties have been accurately determined. Furthermore, understanding (if not complete solution) has emerged even on field theoretic level [51, 53, 56]. Unfortunately, neither of the above processes have been directly related to models of SOC. Accordingly, the present day research is concentrated on absorbing state transitions which have more contact with SOC. An example is the critical point observed in the so called *pair contact process* [57] where particles diffuse only through the birth-death processes given by the reaction scheme

$$A + A \to A + A + A \quad , \quad A + A \to 0 \,, \tag{27}$$

where the first and the second reactions take place with probabilities p and $1-p$, respectively. A related problem is the epidemic model [58] where the reaction scheme

$$A + B \to B + B \quad , \quad B \to A \quad , \quad B0 \leftrightarrow 0B \,, \tag{28}$$

describes static healthy subjects (A) getting infected by diffusing infectious agents (B) who, in turn, recover with time.

The last two models are close to the fixed energy sandpile models (and thus to SOC) in that both of them have an infinite number of absorbing states and their coarse-grained

description involves an order parameter (the active particles) coupled to a static field (the temporarily immobile particles). It has recently been suggested that the similarity may go deeper, i.e. they all belong to the same universality class [59]. This conclusion is based on a field-theoretic calculation near dimension $d = 6$ [59] using Langevin equations which were suggested on phenomenological grounds for the processes (27) and (28) [49, 60]. At this point there is still a debate about both the applicability of the Langevin equations and the validity of the results in lower dimensions. Nevertheless, it appears that the approach of SOC through absorbing state transitions may be coming to an interesting and satisfactory conclusion.

Of course, one should not forget that apart from the connection to SOC, absorbing state transitions in general constitute an important problem in the theory of NESS. The field is developing fast and there are many interesting details scattered across the papers. A guide to the models and to the extensive literature about them can be found in recent reviews [48, 61].

IV.   DISTRIBUTION FUNCTIONS IN NONEQUILIBRIUM STEADY STATES

The simplicity of the description of equilibrium system lies in the existence of the Gibbs distribution i.e. in the elimination of the dynamics from the calculation of averages. Although dynamics is clearly important in nonequilibrium steady states, it is not inconceivable that a prescription exist for a nonequilibrium equivalent of the Gibbs distribution which would include the essential features of the dynamics. Such a distribution function may have singularities as shown in simple examples [62, 63] or it may have problems with the additivity of the associated entropies (which is not unexpected in systems with long-range correlations) [64, 65]. Nevertheless, a prescription with well defined restrictions on its applicability would be valuable and the search for nonequilibrium distribution function(s) has been going on for some time.

A phenomenological approach to the above problem is the non-extensive statistical mechanics [64], an approach that takes its name from the nonextensive character of the postulated entropy. This approach has been much developed during the last decade, and not surprisingly, is has its success in connection with systems which have long-range interactions or display (multi)fractal behavior [64].

Below we shall present a alternative approach that is somewhat less general but it is based on the extention of our knowledge of universality of distribution functions in strongly fluctuating systems.

A.   Power laws and universality of nonequilibrium distributions

Distribution functions of additive quantities such as e.g. the total magnetization in the Ising model are Gaussian in usual equilibrium systems. This Gaussianity follows from the central limit theorem that is applicable due to the correlations being short ranged away from critical points. At critical points, however, the power law correlations result in non-gaussian distributions. The emerging distributions are quite restricted in their possible shapes, however, the reason being that the distribution functions at critical points are scaling functions and their shape is determined by the universality class associated with the given critical point.

The above observation can be used to develop a classification of nonequilibrium distri-

bution functions. Namely, one knows that "effective" criticality (i.e. strong fluctuations and power-law correlations) is the norm for nonequilibrium steady states. Of course, the "effective" critical behavior is determined not only by the interactions but by the dynamics as well. Accordingly, one may expect that the scaling functions (and thus the distribution of macroscopic quantities) are determined by the nonequilibrium universality classes. Once we build a gallery of such scaling functions, we can use them in the same way as in the equilibrium case: We can identify symmetries and underlying mechanisms in experimental systems; we may find seemingly different systems belonging to the same universality class, and thus we can discover common underlying processes present in those systems. We can also use these distribution functions to find the critical dimension of a model and the applications are restricted only by imagination. Below we shall show how to calculate these distribution functions in simple systems and present a few applications.

B.   Picture gallery of scaling functions

The simplest nonequilibrium systems displaying "effective" criticality are the growing surfaces [27, 28]. They are rough quite generally which means that the mean-square fluctuations of the surface diverge with system size. The roughness is defined by

$$w_2 = \frac{1}{A_L} \sum_{\vec{r}} \left[h(\vec{r},t) - \overline{h} \right]^2 \sim L^{\chi} \quad , \tag{29}$$

where A_L is the area of the substrate of characteristic linear dimension L, $\overline{h} = \sum_{\vec{r}} h(\vec{r},t)/A_L$ is the average height of the surface, and χ is a critical exponent characterizing the given universality class. We shall be interested in the steady-state distribution of $P(w_2)dw_2$ and expect that due to criticality, the diverging scale $\langle w_2 \rangle$ will be the only relevant scale in $P(w_2)$ and, consequently, it can be written in a scaling from

$$P_L(w_2) \approx \frac{1}{\langle w_2 \rangle_L} \Phi \left(\frac{w_2}{\langle w_2 \rangle_L} \right) \quad , \tag{30}$$

where $\Phi(x)$ is a scaling function characteristic of the universality class the growth process belong to. Below, we show how to calculate Φ for a simple growth process (Edwards-Wilkinson equation [27]) and will demonstrate that the Φ-s are different for growth processes distinct in the sense of distinct universality classes.

Let us begin by discussing the equations for growing surfaces. In general, deposition of particles on a substrate, under the assumption that the surface formed is a single valued
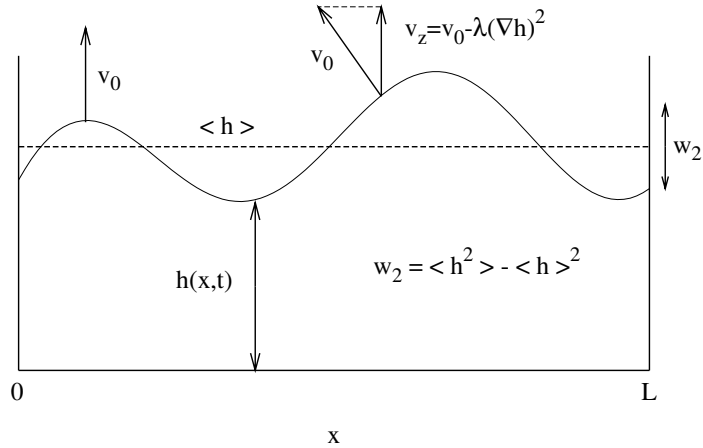
28

FIG. 6: Surface growth. The height of the surface above the substrate is given by $h(x,t)$ and the width of the surface w_2 is characterized by the mean-square fluctuation. The vertical velocity of the surface in general a function of the local properties of the surface, $v(\partial_x h, \partial_x^2 h, ...)$.

function $h(x,t)$, can be described by the equation $\partial_t h = v(h)$ where $v(h)$ gives the velocity of advance of the interface perpendicular to the substrate (Fig.6). The velocity is usually written as $v(h) = v_0 + F(h) + \eta$ where v_0 is the average velocity due to the average rate of deposition, $F(h)$ is related both to the motion of the particles on the surface and to the dependence of the growth on the inclination of the surface. Finally the fluctuations in the above processes are collected in η which is assumed to be a Gaussian white noise in both space and time.

A simple form for $F(h)$ follows from the assumption that particles like to stick at points with large number of neighbors i.e. at large ∂_x^2. Then, $F(h)$ is approximated as $F(h) = \nu\Delta_x^2$ and, in the frame moving with v_0, one has the Edwards-Wilkinson (EW) model [27] of surface growth

$$\partial_t h(x,t) = \nu\Delta h(x,t) + \eta(x,t) . \tag{31}$$

This equation can be solved and one finds that the steady-state probability distribution is given by

$$\mathcal{P}[h(x)] = A e^{-\frac{\sigma}{2}\int_0^L (\nabla h)^2 dx} \tag{32}$$

where σ is related to ν and to the amplitude of the white noise.

Once $\mathcal{P}[h(x)]$ is known, $P(w_2)$ is formally obtained from

$$P(w_2) = \langle\delta(w_2 - [\overline{h^2} - \overline{h}^2])\rangle \tag{33}$$

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where the average $\langle\rangle$ is over all $h(x)$ with the distribution function $\mathcal{P}[h(x)]$ (note that $\overline{h^n}$ is a spatial average and it is still a fluctuating quantity). In practice, it is more convenient to calculate the generating function

$$G(s) = \int_0^\infty e^{-sw_2}P(w_2)dw_2 = \langle e^{-s(\overline{h^2}-\overline{h}^2)}\rangle \tag{34}$$

with the above expression demonstrating why $P(w_2)$ can be calculated analytically in simple models. Namely, if the partition function with $\mathcal{P}$ can be found then the generating function (34) is just the partition function of the model with a quadratic term added and such term usually does not spoil the solvability of the problem. Indeed, e.g. in case of the d=1 EW model with periodic boundary conditions, the problem is reduced to the evaluation of the partition function of a d=1 quantum oscillator thus obtaining [66]

$$G(s) = \prod_{n=1}^\infty \left(1 + \frac{sL}{\sigma\pi^2n^2}\right)^{-1} . \tag{35}$$

Now one can find the average width diverging $\langle w_2 \rangle = -\partial G(s)/\partial s|_{s=0} = L/(6\sigma)$ in the $L \to \infty$ limit. Using $\langle w_2 \rangle$ to eliminate L from (35), one observes that $G(s)$ is a function of the product $s\langle w_2 \rangle$ only and, consequently the inverse Laplace transform yields $P(w_2)$ the scaling form (30). The calculation of the scaling function $\Phi(x)$ consist of collecting contributions from the poles in $G(s)$ and one obtains

$$\Phi(x) = \frac{\pi^2}{3}\sum_{n=1}^\infty (-1)^{n-1}n^2 \exp\left(-\frac{\pi^2}{6}n^2x\right) . \tag{36}$$

Fig.7 shows the above function displaying a characteristic shape of exponential decay $\Phi(x) \sim e^{-\pi^2x/6}$ at large x and essential singularity $\Phi(x) \sim x^{-5/2}e^{-3/(2x)}$ for $x \to 0$.

On Fig.7 we have also included the results for the so called curvature driven growth process which is also called the Mullins-Herring model of surface growth [27]. This is a model where the rearrangement of deposited particles goes on by surface diffusion and the particle current j_h is towards places where there are many neighboring particles i.e. Δh is large. This means that $j_h \sim \nabla\Delta h$ and $F(h) = -\zeta\Delta^2 h$. The resulting equation is called the Mullins-Herring (MH) equation

$$\partial_t h(x,t) = -\zeta\Delta^2 h(x,t) + \eta(x,t). \tag{37}$$

The surfaces described by the MH equation belong to a universality class distinct from that of the EW growth. Indeed, the MH equation can be solved easily and one finds that
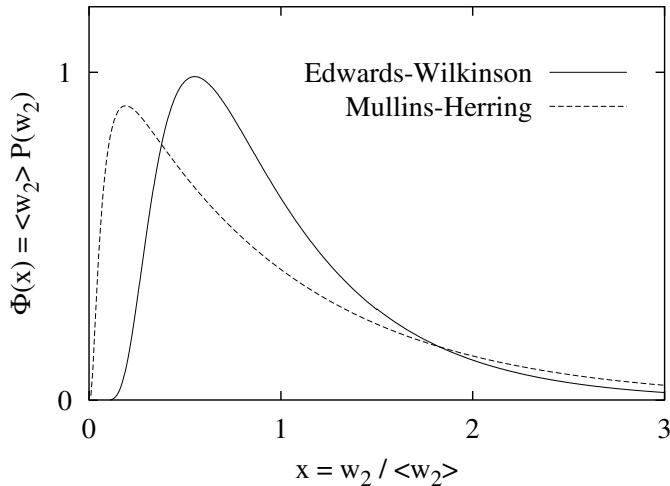
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FIG. 7: Comparison of the scaling functions for the EW and MH models.

$\langle w_2 \rangle_{MH} \sim L^2$ in contrast to the EW result $\langle w_2 \rangle_{EW} \sim L$. Accordingly, the scaling function should also be different. A calculation similar to that described above for the EW case verifies this expectation [67] as can be observed on Fig.7.

An important point to remark about the comparisons of the EW and MH curves is that they are well distinguishable. Their maximum, their small x cutoff, and their decay at large x are all sufficiently different so that no ambiguity would arise when analyzing experimental data. Indeed, the $d = 2$ versions of the above Φ-s as well as a number of others characterizing various growth processes have been obtained in [68] and it did not appear to be difficult to pick the scaling function which was corresponding to a given set of experiments [68].

Finding out the universality class of a growth process is one possible application if one has a sufficiently developed gallery of scaling functions. Below we discuss other possibilities for application.

C. Upper critical dimension of the KPZ equation

The KPZ equation [29] is the simplest nonlinear model describing growth in terms of a moving interface. It differs from the EW model by taking into account that the surface grows in the direction of its normal provided the incoming particles have no anisotropy in their arrival direction. Then the z component of the velocity of the surface has a correction term proportional to $(\nabla h)^2$ as shown on Fig.6 and the equation in lowest order in the

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nonlinearities becomes the so called KPZ equation

$$\partial_t h = \nu\vec{\nabla}^2 h + \lambda(\vec{\nabla}h)^2 + \eta\,. \tag{38}$$

Here ν and λ are parameters, while $\eta(\vec{r},t)$ is again a Gaussian white noise. The steady state surfaces generated by (38) appear to be rough (critical) in any dimension if the nonequilibrium drive (λ) is large enough. Since eq.(38) gives account of a number of interesting phenomena (Burgers turbulence, directed polymers in random media, etc.) lots of efforts have been spent on finding and understanding the scaling properties of its solutions [27, 28]. Nevertheless, a number of unsolved issues remain, the question of upper critical dimension (d_u) being the most controversial one. On one hand mode-coupling and other phenomenological theories suggest that $d_u = 4$ [69] while all the numerical work fail to find a finite d_u and the indication is that $d_u = \infty$ [70]. Below I would like to show how the scaling functions of the roughness can shed some light on this controversy [71].

Let us begin with the observation that scaling functions do not change above d_u. Thus if we build $\Phi(x)$ in dimensions $d = 1 - 5$ and observe that they differ significantly in $d = 4$ and 5 then we can conclude that $d_u > 4$. Since $\Phi(x)$ cannot be exactly calculated for $d \geq 2$ we must evaluate it through simulations with the results displayed on Fig.8.
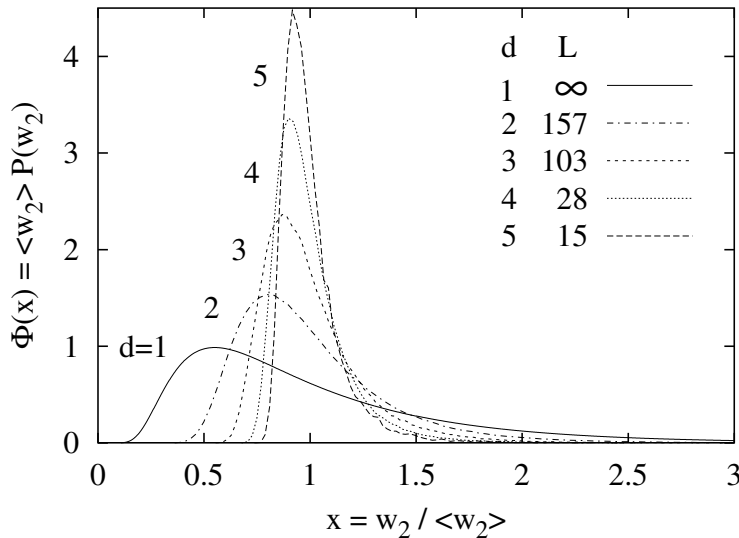

FIG. 8: Roughness distribution for KPZ steady-state surfaces in dimensions $d = 1 - 5$.

As one can see on Fig.8, the scaling functions change smoothly with d. The $\Phi(x)$-s get narrower and more centered on $x = 1$ with increasing d, and there is no break in this

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behavior at $d = 4$. The equality of the d=4 and 5 scaling functions appears to be excluded.

Of course, our result is coming from numerical work with the same general conclusion as in previous studies. So, why is it more believable? Because one of the main criticism of numerical studies does not apply to it. Namely, no fitting parameters and fitting procedures are used in contrast to the usual determination of critical exponents. One just builds the histograms, calculates the averages to determine the scaling variable and plots the scaled histogram. This is clear and well understood but there is another remarkable feature in these scaling functions the origin of which is less obvious. Namely, the scaling functions in principle should depend on the size of the system,

$$\Phi(x) = \Phi_L(w_2/\langle w_2 \rangle_L).\tag{39}$$

What is observed, however, is that the L dependence is practically all in $\langle w_2 \rangle_L$ and L dependence of the shape of Φ (explicit dependence on L) disappears already at small L. These are important points and the KPZ application of the scaling functions was mainly chosen to emphasize them.

It seems that these scaling functions are versatile tools which can be used in computer science [72] as well as in understanding the propagation of chemical fronts [73]. An interesting application was e.g. the establishment of a connection between the energy fluctuations in a turbulence experiment and the interface fluctuations in the $d = 2$ Edwards-Wilkinson model [74], and thus prompting a search for an interface interpretation of the dissipative structure in the turbulent system [75]. In another case, it helped to make a link between the much studied $1/f$ noise and the extreme value statistics [76]. Since the effective criticality is a real feature of many nonequilibrium system, we expect that many more use will be found for the scaling functions discussed in this section.

V. QUANTUM PHASE TRANSITIONS

Quantum critical points are associated with the change of the symmetry of the ground state of a quantum system as the interactions or an external field (control parameter) are varied. They have been much investigated in recent years with the motivation coming from solid state physics [78]. Namely, the strongly-correlated electron systems often produce power law correlations, and the origin of the observed scale-invariance is suggested to be the presence of a quantum critical point at $T = 0$ provided the effect of the quantum phase transition is felt at finite T as well.

From our point of view, the quantum phase transitions are interesting because they are good candidates for studying the effects of a nonequilibrium drive on well established symmetry-breaking transitions. The advantage of these systems is that there is no arbitrariness in their dynamics (it is given by quantum mechanics), the one-dimensional systems are simple with examples of exactly solvable models displaying genuine critical phase transitions (see e.g. the transverse Ising chains discussed below) and, furthermore, there is much previous work to build on.

The only problem is how to force a quantum system into a non-equilibrium steady state. An obvious way is to attach two heat baths of different temperatures at the two ends of a spin chain. Unfortunately, this makes the problem unsolvable (even numerically) for any reasonable size system [79] and thus it is practically impossible e.g. to draw conclusions about the long-range correlations generated in the system. Below we show a way to avoid the problem of heat baths. The idea is that the nonequilibrium steady states always carry some flux (of energy, particle, momentum, etc.). Thus a steady state that is presumably not very far from the one generated by boundary conditions may be constructed by constraining the quantum system to have a flux equal to the one generated by the boundary conditions. For example, in the case of the transverse Ising chain treated below, we shall constrain the system to carry an energy current and will investigate the correlations in this constrained state.[1]

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[1] It should be noted that this section was not discussed during the main lectures of the course in Les Houches. It was described only in a seminar for interested students.

A. Spin chains with fluxes

As a simple model with critical phase transition, we consider the $d = 1$ Ising model in a transverse field h which has the following Hamiltonian:

$$\hat{H}_I = -\sum_{\ell=1}^{N} \left(\sigma_\ell^x \sigma_{\ell+1}^x + \frac{h}{2}\sigma_\ell^z \right) . \tag{40}$$

Here the spins σ_ℓ^α ($\alpha = x, y, z$) are represented by $1/2$ times the Pauli matrices located at the sites $\ell = 1, 2, ..., N$ of a one-dimensional periodic chain ($\sigma_{N+1}^\alpha = \sigma_1^\alpha$). The transverse field, h, is measured in units of the Ising coupling, J, which is set to $J = 1$ in the following.

This model can be solved exactly [84, 85] and it is known that a second order phase transition takes place in the system as h is decreased. The order parameter is the expectation value $\langle \sigma_x \rangle$ i.e. $\langle \sigma_x \rangle = 0$ for $h > 1$, while $\langle \sigma_x \rangle \neq 0$ for $h < 1$ and $h_c = 1$ is a critical point. The scaling behavior at and near h_c belongs to the $d = 2$ Ising universality class.

In order to constrain the above system to carry a given energy flux J_E we shall use the Lagrange multiplier method. Namely, we add a term $\lambda \hat{J}_E$ to the Hamiltonian where $\hat{J}_E$ is the local energy flux operator summed over all sites, and find that value of λ which produces a ground state with the expectation value $\langle \hat{J}_E \rangle = J_E$.

The above scheme requires the knowledge of the local energy current, $\hat{J}_\ell$. It can be obtained using the quantum mechanical equation of motion for the energy density $\dot{\varepsilon}_\ell = i/\hbar[\hat{H}_I, \varepsilon_\ell]$, and representing the result as a divergence of the energy current $\dot{\varepsilon}_\ell = J_\ell - J_{\ell+1}$. The calculation yields ($\hbar = 1$ is used in the following)

$$\hat{J}_\ell = \frac{h}{4}\sigma_\ell^y(\sigma_{\ell-1}^x - \sigma_{\ell+1}^x) \tag{41}$$

and this allows to construct the 'macroscopic' current $\hat{J}_E = \sum_\ell \hat{J}_\ell$. Adding it to $\hat{H}_I$ with a Lagrange multiplier, $-\lambda$,

$$\hat{H} = \hat{H}_I - \lambda \hat{J}_E \quad . \tag{42}$$

we obtain the Hamiltonian whose ground states with $\langle \hat{J}_E \rangle = J_E \neq 0$ will give us information about the current carrying states of $\hat{H}_I$.

In order to avoid confusion, we emphasize that the energy current, $\hat{J}_E$, is associated with $\hat{H}_I$ and not with the new Hamiltonian, $\hat{H}$. We also note that $\hat{H}$ is just another equilibrium Hamiltonian, it differs from $\hat{H}_I$ by an extra term which breaks the left-right symmetry of $\hat{H}_I$. Finding the ground state of $\hat{H}$, however, gives us the minimum energy state of $\hat{H}_I$ which

carries an energy current, $J_E = \langle \hat{J}_E \rangle$. Thus the ground-state properties of $\hat{H}$ provide us with the properties of the nonequilibrium steady states of the transverse Ising model.

It turns out that $[\hat{H}_I, \hat{J}_E] = 0$ and $\hat{H}$ can be diagonalized by the same transformations which diagonalize $\hat{H}_I$ [80], and one arrives to a system of free fermions with a spectrum of excitation energies given by $\omega_q = |\Lambda_q|$ where

$$\Lambda_q = \frac{1}{2}\sqrt{1 + h^2 + 2h\cos q} + \frac{\lambda h}{4}\sin q \quad . \tag{43}$$

with the wave numbers restricted to $-\pi \leq q \leq \pi$ in the thermodynamic limit $(N \to \infty)$.
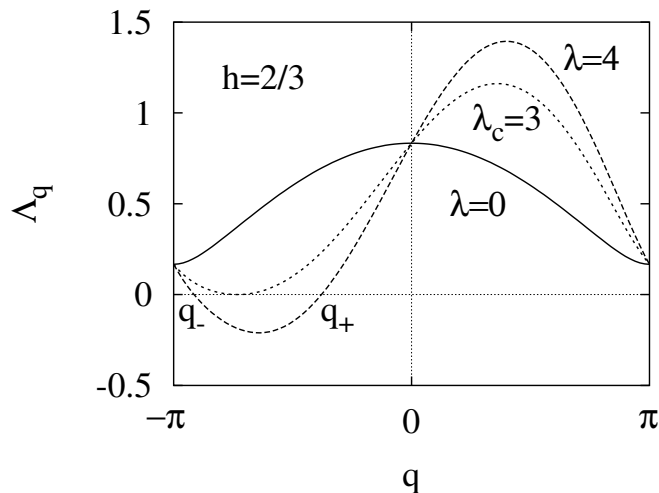

FIG. 9: Spectrum of the transverse Ising model in the presence of a field (λ) which drives the current of energy. The excitation energies are given as $\omega_q = |\Lambda_q|$. Increasing the drive makes the ground-state change at a critical $\lambda = \lambda_c$ ($\lambda_c = 3$ for $h = 2/3$) when negative energy states start to appear and get occupied. The qualitative picture is the same at all transverse fields h.

Fig.9 displays the spectrum for $h = 2/3$ and various λ and one can see that the $q \to -q$ symmetry of the spectrum is broken for $\lambda \neq 0$. Nevertheless, for small λ the ground-state remains that of the transverse Ising model ($\lambda = 0$) since $\Lambda_q \geq 0$ and the occupation number representation of the ground state does not change. Accordingly, no energy current flows ($J_E = 0$) for $\lambda < \lambda_c$. This rigidity of the ground state against the symmetry-breaking field which drives the energy current is a consequence of the facts that the fermionic spectrum of the transverse Ising model has a gap and that the operator $\hat{J}_E$ commutes with $\hat{H}_I$ (similar rigidity is observed in the studies of energy flux through transverse XX chain [82]).

The ground-state properties do change when $\Lambda_q < 0$ in an interval $[q_-, q_+]$ and these q
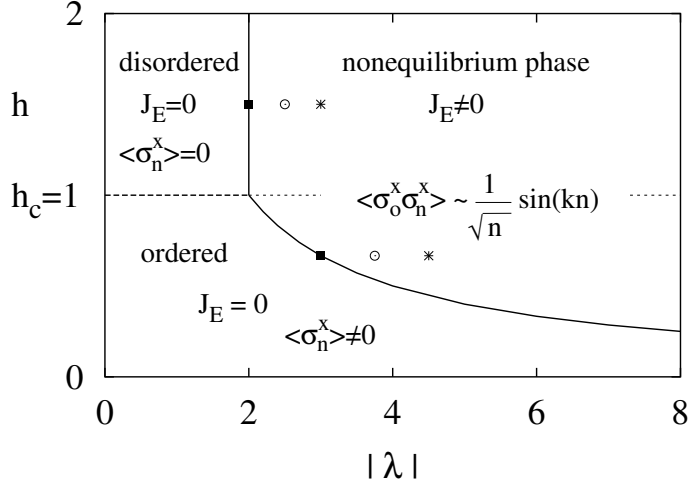
FIG. 10: Phase diagram of the driven transverse Ising model in the $h - \lambda$ plane where h is the transverse field while λ is the effective field which drives the flux of energy. Power-law correlations are present in the nonequilibrium phase ($J_E \neq 0$) and on the Ising critical line in the equilibrium phase ($J_E = 0$, dashed line).

states become occupied. Due to the resulting asymmetry in the occupation of the q and $-q$ states, the energy current becomes nonzero. The line $\lambda_c(h)$ which separates the region of unchanged transverse Ising behavior from the $J_E \neq 0$ region is obtained from the conditions $\Lambda_q = 0$ and $\partial \Lambda_q / \partial q = 0$, and is displayed on the phase diagram (Fig.10) as a solid line. Another phase boundary on Fig.10 is shown by dashed line. It separates the magnetically ordered ($h < 1$, $\lambda < 2/h$) and disordered ($h \geq 1$, $\lambda < 2$) transverse Ising regions. Since the ground state is independent of λ for $\lambda < \lambda_c$, one has the same second order transition across the dashed line as at $h = 1$ and $\lambda = 0$ i.e. it belongs to the $d = 2$ Ising universality class [85].

Clearly, one can view the region $\lambda < \lambda_c$ as an equilibrium phase while the $\lambda > \lambda_c$ region as a nonequilibrium one since there is a nonzero energy flux through the latter. This flux can actually be calculated easily with the simple result

$$j_E = \langle \hat{J}_E / N \rangle = (4\pi)^{-1} \sqrt{(1 - 4/\lambda^2)(h^2 - 4/\lambda^2)} \quad . \tag{44}$$

Apart from the fact that $J_E \neq 0$, the $\lambda > \lambda_c$ region should also be considered as a distinct phase since the long-range magnetic order existing for $h < 1$ breaks down when $J_E \neq 0$ and the magnetic correlations become oscillatory with amplitudes decaying as a power of distance. Indeed, this can be seen by investigating the $\langle \sigma_\ell^x \sigma_{\ell+n}^x \rangle$ correlations which

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can be expressed through Pfaffians [81] and thus making possible numerical calculations for $n \leq 100$. In the presence of long-range order one should have $\langle \sigma_\ell^x \sigma_{\ell+n}^x \rangle \to \langle \sigma_\ell^x \rangle^2 \neq 0$ for $n \to \infty$ while we find that the correlations decay to zero at large distances as

$$\langle \sigma_\ell^x \sigma_{\ell+n}^x \rangle \sim \frac{Q(h,\zeta)}{\sqrt{n}} \cos(kn) \qquad (45)$$

where the wavenumber, $k = \arccos(2/\lambda h)$. The above result (45) is coming from numerics and it is exact in the $\lambda \to \infty$ limit where the correlations are those of the $d = 1$ XX model [81].

One can observe power-law correlations for $\lambda > \lambda_c$ in other physical quantities as well. For example, the envelopes of both $\langle \sigma_\ell^z \sigma_{\ell+n}^z \rangle$ and $\langle \hat{J}_\ell \hat{J}_{\ell+n} \rangle$ correlations behave as n^{-2} in the large n limit [80]. Thus we arrive to the main conclusion of this section, namely that a simple, exactly soluble quantum system shows power-law correlations in the current carrying state in agreement with the notion that power-law correlations are a ubiquitous feature of nonequilibrium steady states.

Actually, remembering that power-law correlations in quantum models are associated with a gapless excitation spectrum, we can reformulate the transverse Ising model result to see a general connection between the emergence of power-law correlations and the presence of a current. Indeed, let us assume that a system with Hamiltonian $\hat{H}_0$ has a spectrum with a gap between the ground-state and the lowest excited state. Furthermore, let $\hat{J}$ be a 'macroscopic' current of a conserved quantity such that $[\hat{H}_0, \hat{J}] = 0$. Generally, there is no current in the ground state and adding $-\lambda \hat{J}$ to $\hat{H}_0$ does not change the $\langle \hat{J} \rangle = 0$ result for small λ. Current can flow only if some excited states mix with the ground state and, consequently, a branch of the excitation spectrum must come down and intersect the ground-state energy in order to have $\langle \hat{J} \rangle \neq 0$. Once this happens, however, the gap disappears and one can expect power-law correlations in the current-carrying state. Admittedly, the above argument is not strict and is just a reformulation (in general terms) of what we learned from the transverse Ising model. We believe, however, that the above picture is robust and suggestive enough to try to find other soluble examples displaying the flux $\to$ power-law-correlations relationship.

B. Quantum effective interactions

When using the Lagrange multiplier method, one assumes that the flux generated by boundary conditions can be replaced by the effective interactions contained in an appropriately chosen global flux $\hat{J}$. These interactions are generally short ranged since the flux is usually a sum of local terms. The short-range nature of the effective interactions is actually not in contradiction with the power-law correlation being generated since the Lagrange multiplier gets tuned in order to achieve a given flux and, furthermore, the tuning is not quite trivial since λ must be increased past a critical value in order to have e.g. a nonzero flux energy. The question nevertheless arises whether adding a flux term in $\hat{H}$ was an adequate description for the nonequilibrium steady state which is expected to display power-law correlations.

In order to investigate the above question one would have to solve the problem with the boundary drive but, as discussed above, an exact solution does not seem to be feasible. Instead, however, one can prepare initial conditions which will lead in the long-time limit to a steady flux. Then the steady state obtained in this natural way can be compared to the one found by the Lagrange multiplier method. This program has been carried out [83] for the XX chain defined by the following Hamiltonian

$$\hat{H}_{XX} = -\sum_{\ell=1}^{N} \left(\sigma_\ell^x \sigma_{\ell+1}^x + \sigma_\ell^y \sigma_{\ell+1}^y + h\sigma_\ell^z \right) . \tag{46}$$

In this model, the transverse magnetization $M_z = \sum_i \sigma_i^z$ is conserved and one can investigate the nonequilibrium states which carry a given magnetization flux by using the Lagrange multiplier method [82]. At the same time, the model is simple enough so that one can solve the time dependent problem where a steady magnetization flux is achieved by starting with an inhomogeneous initial state that is the ground state at fixed magnetization but with $m = \langle s_n^z \rangle$ reversed from m_0 for $n \leq 0$ to $-m_0$ for $n > 0$. The time-evolution of this step-like initial state can be followed exactly and the magnetization profile emerging in the long-time limit is shown on Fig.11. The remarkable feature of this magnetization profile is the middle part which is an $m = 0$ homogeneous state carrying a magnetization flux $j(m_0)$. Comparing this state to the one generated by adding the flux term to the Hamiltonian and fixing the Lagrange multiplier to have the same $j(m_0)$, we find that various expectation values such as the energy, the occupation number in fermionic representation are all equal in the two
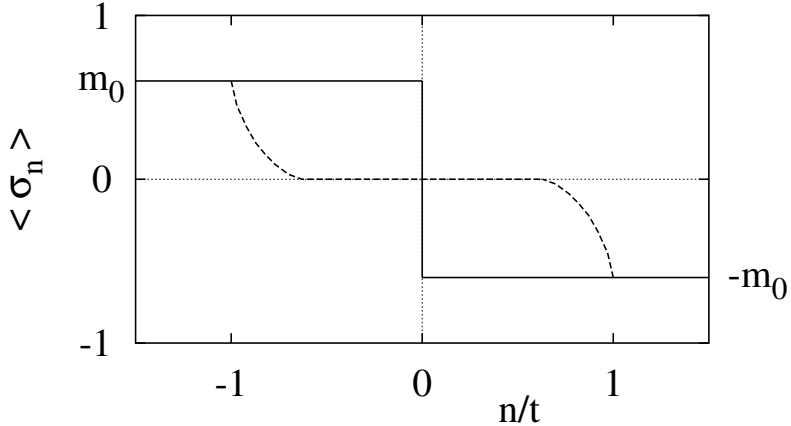
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FIG. 11: Time evolution of the magnetization profile starting from a step-like initial conditions shown as solid line. There are two fronts going out to $\pm\infty$. They diminish the magnetization and leave behind a homogeneous $\langle s_n^z \rangle = 0$ state. In the scaling limit $t \to \infty$, $n/t \to x$, the magnetization $m(n,t) \approx \Phi(n/t)$ is given by $\Phi(x) = m_0$ for $-1 < x$, $\Phi(x) = m_0 - \pi^{-1}\arccos(x)$ for $-1 < x < -\cos(\pi m_0)$, $\Phi(x) = 0$ for $-\cos(\pi m_0) < x < 0$, and $\Phi(x) = -\Phi(-x)$ for $x > 0$ (dashed line).

states. Thus the Lagrange multiplier yields a correct description of the states carrying a magnetization flux.

It should be noted, however, that recent calculation with an inhomogeneous initial state of different temperatures (T_1 for $x < 0$ and T_2 for $x > 0$) has yielded a different result for the asymptotic state carrying an *energy* flux [86]. Namely, it was shown that, at least in the neighborhood of $x = 0$ and in the $t \to \infty$ limit, the properties of the flux-carrying state can be interpreted in terms of the ground state of an effective Hamiltonian

$$\hat{H}_{\text{eff}} = \hat{H}_{XX} + \sum_{n=1}^{N} \mu_n \sum_{j=1}^{N} \hat{Q}_j^{(n)}, \tag{47}$$

where $Q_j^{(n)}$ is a product of local operators at sites j and $j + n$, and the interaction is of long-range type since $\mu_n \sim 1/n$ [86] (remarkably, the first two operators $\hat{Q}_j^{(1)}$ and $\hat{Q}_j^{(2)}$ are those appearing in the Lagrange multiplier treatment of the energy flux in the XX chain). Although the homogeneity of the asymptotic state was not shown and thus the comparison may be questioned, the above result indicates that the Lagrange multiplier approach may be only a first approximation in describing the flux-carrying states.

In summary, the studies of quantum systems described above strengthen the view that

fluxes generate long-range correlations. Furthermore, the quantum systems also give a simple picture of how the emergence of these correlations is related to the closing a gap in the excitation spectrum above the ground state.

VI. OUTLOOK

There are topics which are important but were not discussed in these lectures. To mention a few, there is a large body of work on one-dimensional systems displaying nonequilibrium phase transitions, on orderings of granular gases under shear, on pattern formation with phase transitions described by the complex-coefficient Landau-Ginzburg equation, and the list could be continued. My choice of topics mainly reflects my past work and my attempts to develop simple starting points for making inroads into the beautiful but rather difficult field of far from equilibrium phenomena.

Finally, I was asked to provide entertainment for readers by trying to guess the future developments in connection with nonequilibrium orderings. Well, one of the present problem of the field is the lack of simple experimental systems which can be be taken far enough from equilibrium and compared to elementary models of NESS. Search for such systems will intensify and I expect that there will be a shift towards biological problems. There the condition of being far from equilibrium is satisfied and there may be surprisingly simple phenomena under the guise of complicated pictures. This line of research may in the future meet up with game theories generalized to take into account spatial structures.

Search for better understanding of the emerging effective interactions will also continue, just as the sorting out of the absorbing-state transitions (surprising connections may be still found there, in addition the existing one to SOC). I also believe that theory of nonequilibrium distributions will be much developed, and limiting distributions such as the ones emerging in extreme statistics will have a much wider use in physics. These are more or less safe bets. And then there is the unpredictable part of future.

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