Nonequilibrium Phase Transitions

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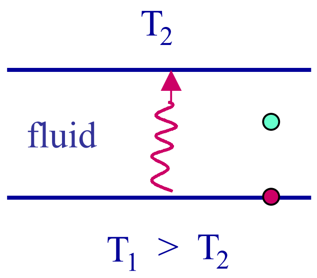
Outline

(1) Nonequilibrium steady states

Breaking of detailed balance ⇒ problems with usual thermodynamic concepts
Presence of fluxes ⇒ power-law correlations, anisotropy

(2) Phase transitions far from equilibrium

Differences from equilibrium (effects from dynamics)
Generation of effective long-range interactions, dynamical anisotropies
Searching for nonequilibrium universality classes
Driven lattice gases, two-temperature models, flocking, …
Where do the power-law correlations come from?
SOC and absorbing-state transitions, surface fluctuations
Nontrivial distribution functions - using universality



(3) Quantum steady-states with fluxes

Spin chains with fluxes: T=0 nonequilibrium transitions

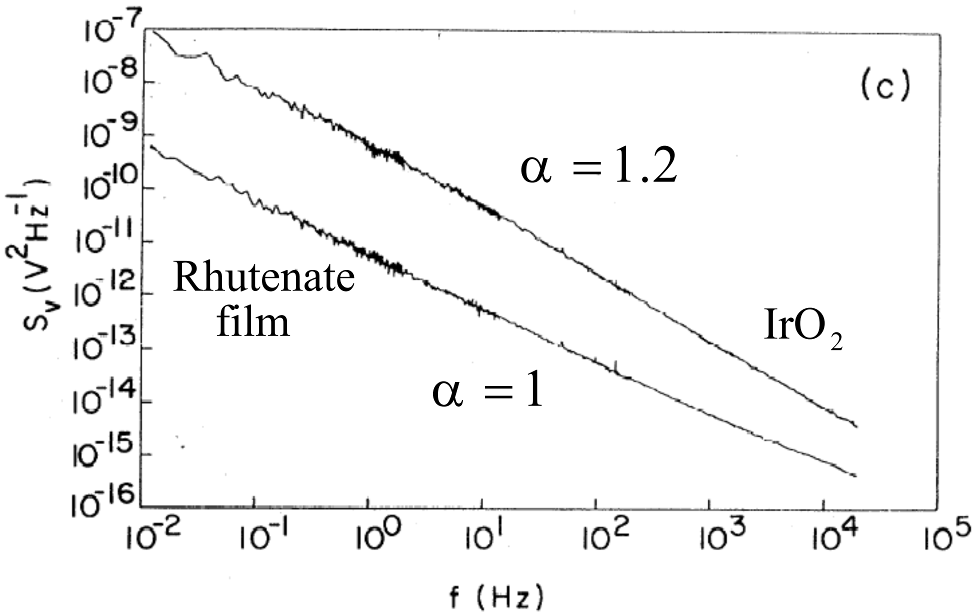
(4) Pattern formation

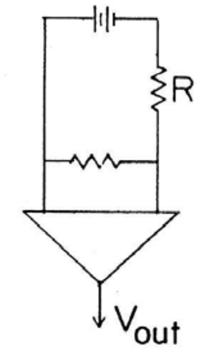
Classification of instabilities
Real- and complex-coefficient Ginzburg-Landau equations

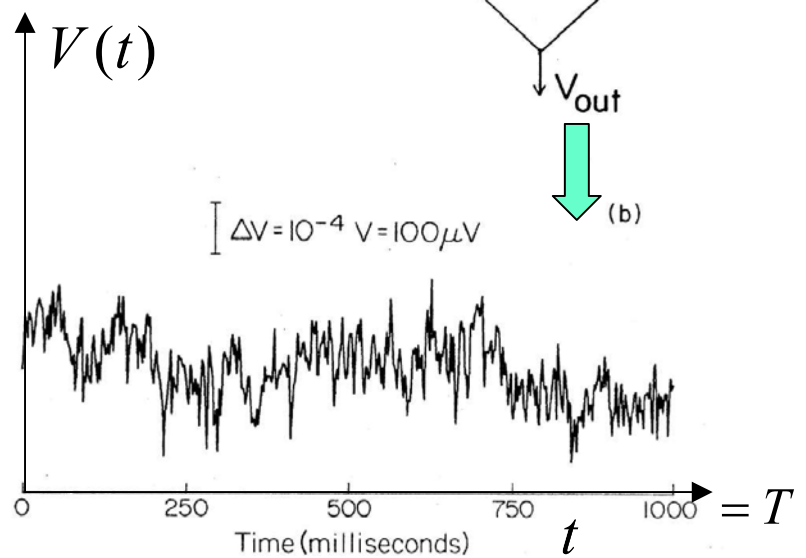
Lecture notes:
http://mpej.unige.ch/homepage/courses/

Where do the power laws come from?

History - examples 1/f noise - voltage fluctuations in resistors





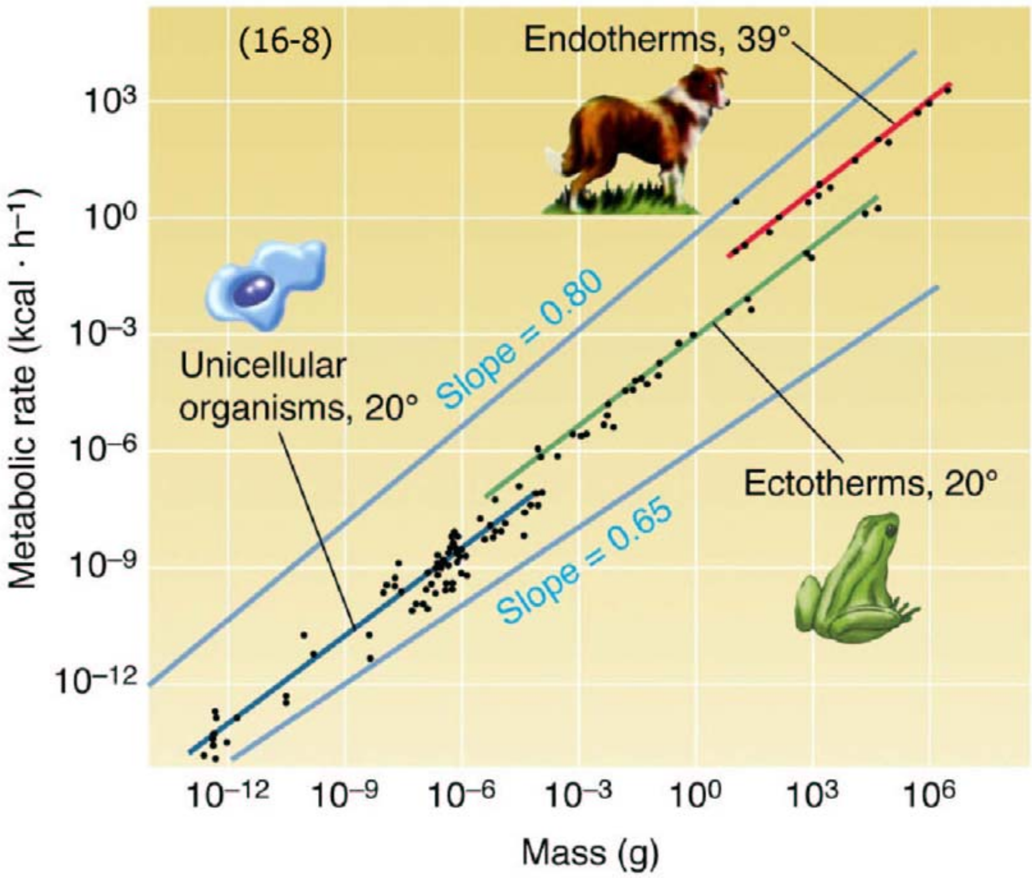


$$S(f) \sim \langle |V_f|^2 \rangle \sim 1/f^{\alpha}$$

$$V_f = \tfrac{1}{T} \sum_x e^{2\pi i f t} V(t)$$

M.B. Weissman, Rev.Mod.Phys.60, 537(1988)

Metabolic rate vs. mass of animals



Where do the power laws come from?

Other examples:

Earthquakes,

turn-around-time fluctuations in internet

income distributions,   light emission in white-dwarfs, ……...

fluctuations in sand flow,

Basic questions:

Are power laws necessarily related to a phase transition (critical behavior)?

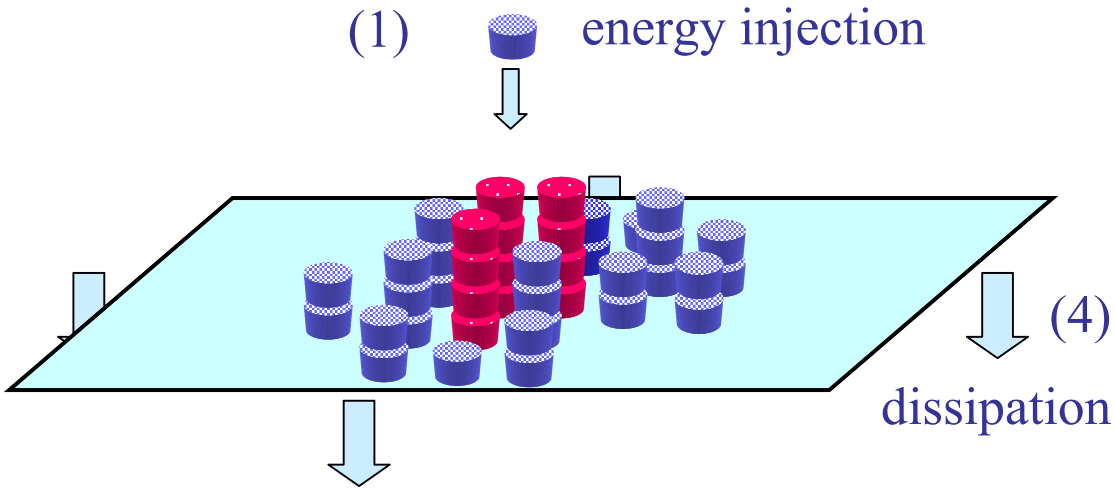
Can 1/f noise (scaling) be the natural outcome of a complex dynamics?

What ingredients determine the characteristics of the power laws?

SOC

Bak-Tang-Wiesenfeld (1987)

Self Organized Criticality = Criticality without tuning



(2)

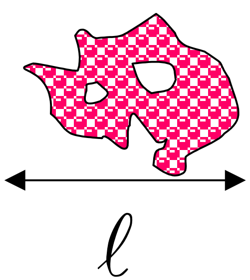
$$z_{crit} = 4$$

(3)

$$z_i > 4 \rightarrow z_i - 4$$

redistribution to neighbors

Avalanche (sequence of topplings between two additions)



Size-distribution of avalanches

$$\boxed{P(\ell) \sim \ell^{-\tau_\ell}}$$

Lifetime-distribution of avalanches

$$\boxed{P(t) \sim t^{-\tau_t}}$$

no fine tuning of parameters

SOC

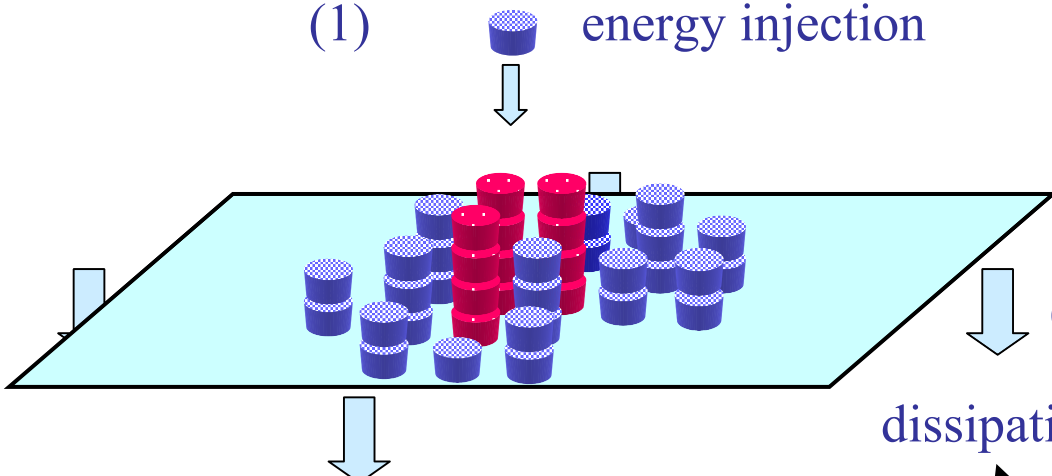
Is it criticality without tuning?

Nonlocal dynamics (external supervisor)

(1) energy injection

(2)

$$z_{crit} = 4$$



(3)

$$z_i > 4 \rightarrow z_i - 4$$

(4)

dissipation

redistribution to neighbors

local dynamics

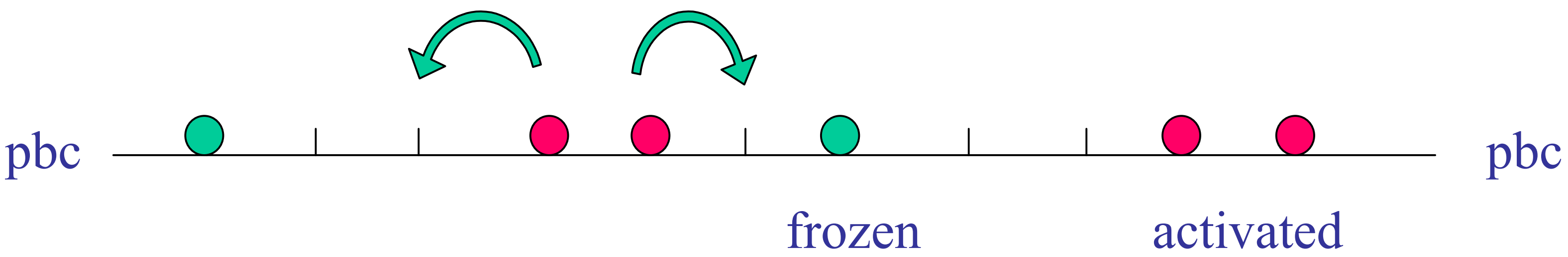
Separation of timescales

Source (energy injection) and sink (dissipation) is infinitesimal

SOC and absorbing-state transitions

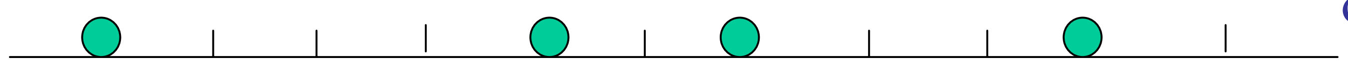
Dickman, Munoz, Vespignani, Zapperi (1998)

Activated random walkers ($\longrightarrow$ fixed energy sandpiles)



Absorbing states

infinite number of them



Particle conservation $\longrightarrow$ Control parameter : n=N/L $\longrightarrow$ $\boxed{n_c = 1/2}$

active - frozen transition

Order parameter: Number of n.n. pairs (activity)

Order parameter is coupled to a static field

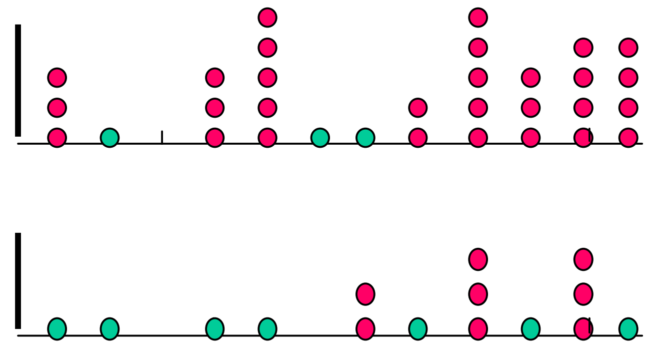
SOC as pinching of a critical point

Fixed energy sandpile with closed boundary

Conserved total energy E - contr. par.

Ever lasting activity for $E > E_c$

Dying out activity for $E < E_c$

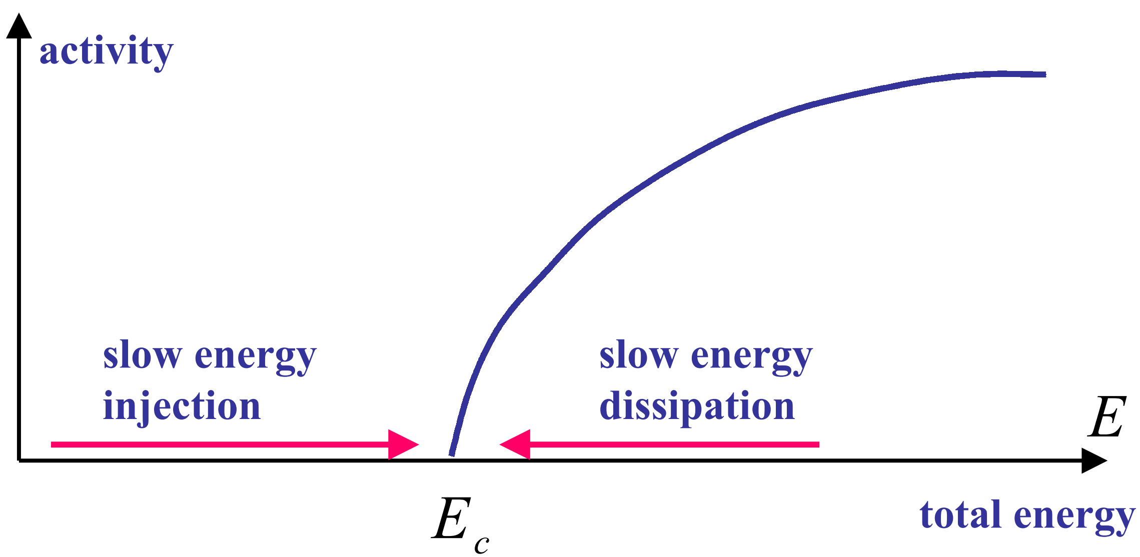


$z_c = 2$

Order parameter: activity

Switch on dissipation
(at boundary)

Switch on injection
(if activity is ceased)



Program:

Calculate the exponents of SOC from the appropriate absorbing state transition

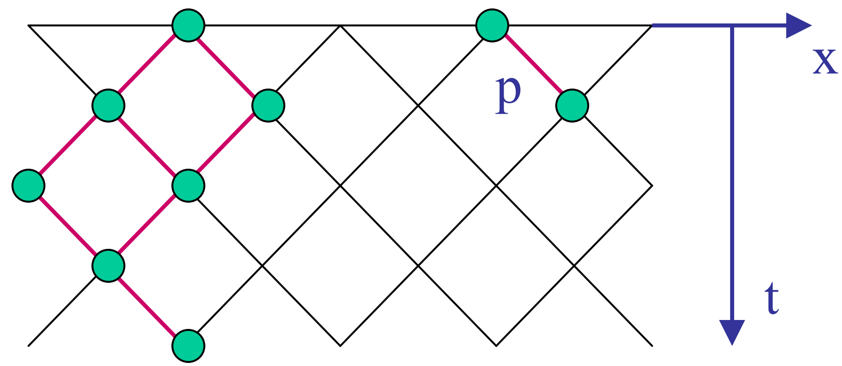
Examples of absorbing-state transitions I

Directed percolation (DP)

Control parameter: p

Order parameter:

$$\langle \rho \rangle = \rho(t \to \infty)$$



single absorbing state

Exponents:

$$\langle \rho \rangle \sim (p - p_c)^{\beta}$$

d=1 $\beta = 0.276$

Reaction-diffusion equivalent: ● $\equiv$ $\mathbf{A}$

Death $\mathbf{A} \to 0$ **Diffusion** $0\mathbf{A} \to \mathbf{A}0$

Birth $\mathbf{A} \to 2\mathbf{A}$ **Coagulation** $2\mathbf{A} \to \mathbf{A}$

Field theory: $d_c=4$

$$\partial_t \rho = \Delta\rho + r\rho - \rho^2 + \zeta$$

New (and difficult): multiplicative noise

$$\langle \zeta(x,t)\zeta(x',t')\rangle = c(\rho)\delta_{x,x'}\delta_{t,t'}$$

in Landau's spirit $c(\rho) \sim \rho$

Early claims: everything is DP.
Now: single absorbing state, no symmetries, no conservation laws, short-range dynamics

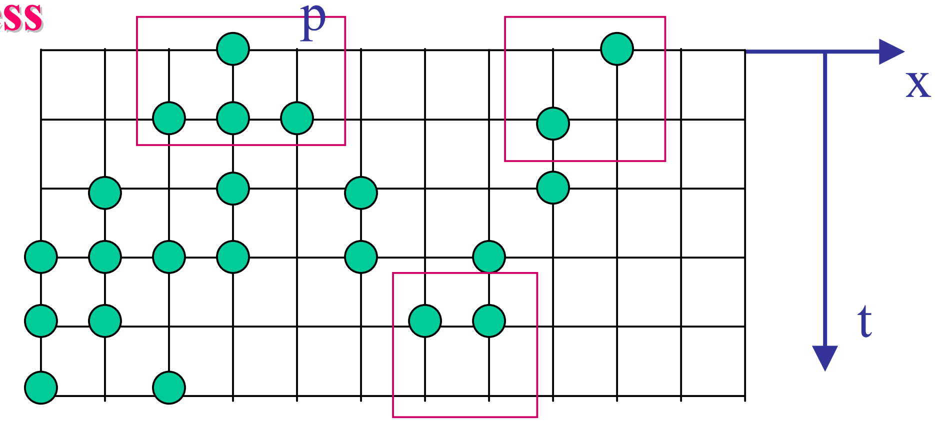
Examples of absorbing-state transitions II

Parity conserving (PC) process

Control parameter: p

Order parameter:

$$\langle \rho \rangle = \rho(t \to \infty)$$



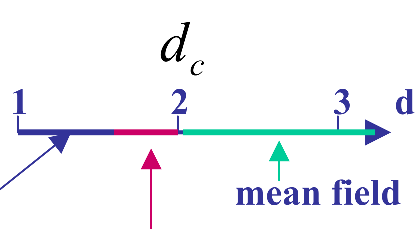
two absorbing states

Exponents:

$$\langle \rho \rangle \sim (p - p_c)^{\beta}$$

d=1 $\beta = 0.94$

Field theory: Complex noise



$p_c \neq 0$

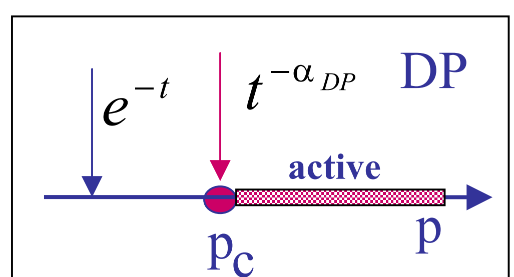
$p_c = 0 \;\; \langle \rho \rangle \sim p^{\beta'}$

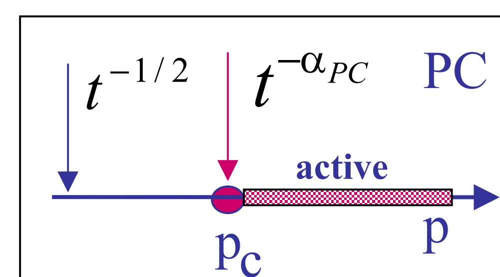
$p_c = 0$
$\langle \rho \rangle \sim p$

Reaction-diffusion equivalent: ● $\equiv$ **A**

Annihilation $2\mathbf{A} \to 0$ **Birth** $\mathbf{A} \to 3\mathbf{A}$

Diffusion $0\mathbf{A} \to \mathbf{A}0$





Examples of absorbing-state transitions III

Pair Contact Process with Diffusion (PCPD)

Control parameters: p, D

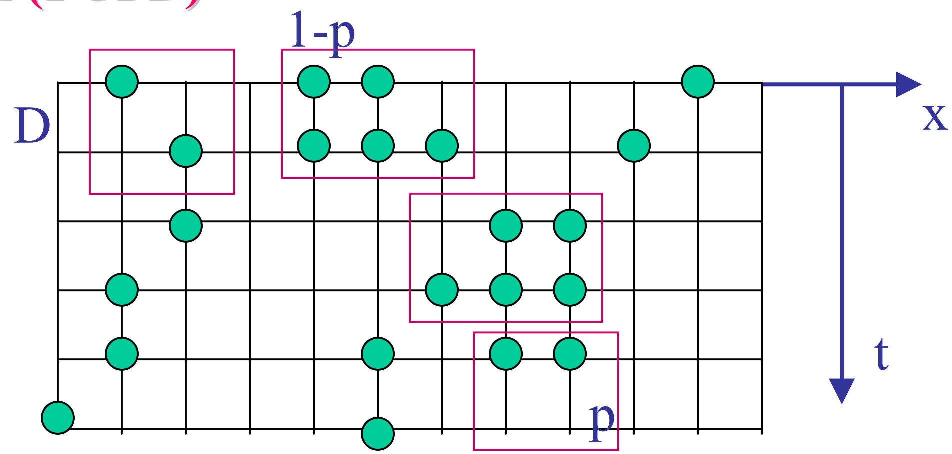
Order parameter:

$$\boxed{\langle \rho \rangle = \rho(t \to \infty)}$$

Exponents:

$$\boxed{\langle \rho \rangle \sim (p - p_c)^{\beta}}$$

d=1 $\boxed{\beta = 0.58}$



Reaction-diffusion equivalent: $\bullet \equiv \boxed{\mathbf{A}}$

Annihilation $\boxed{2\mathbf{A} \to 0}$ **Birth** $\boxed{2\mathbf{A} \to 3\mathbf{A}}$

Diffusion $\boxed{0\mathbf{A} \to \mathbf{A}0}$

Two absorbing states (0 or 1 particle) - no conservation law

D=0 : infinite number of abs. states,
PD class except spreading

Field theory:?

G. Ódor, cond-mat/0205644

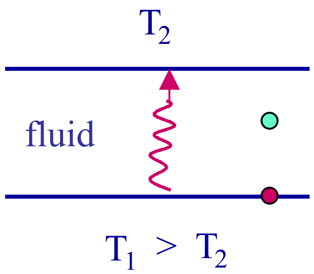
F. van Wijland et al., cond-mat/0312463

Distribution Functions for Nonequilibrium Fluctuations

(universality, extreme statistics, etc.)

Question: What is the probability distribution function (PDF) of a macroscopic quantity in a far from equilibrium system?

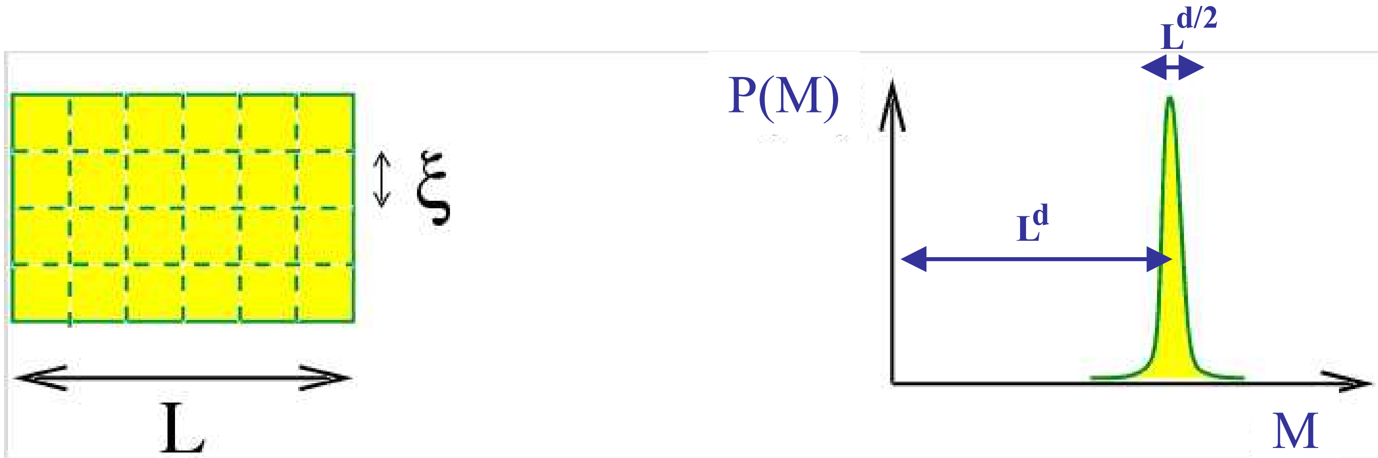
Idea: Effective criticality (power-law correlations) may imply
(1) nongaussian PDF-s,
(2) universal forms according to universality classes



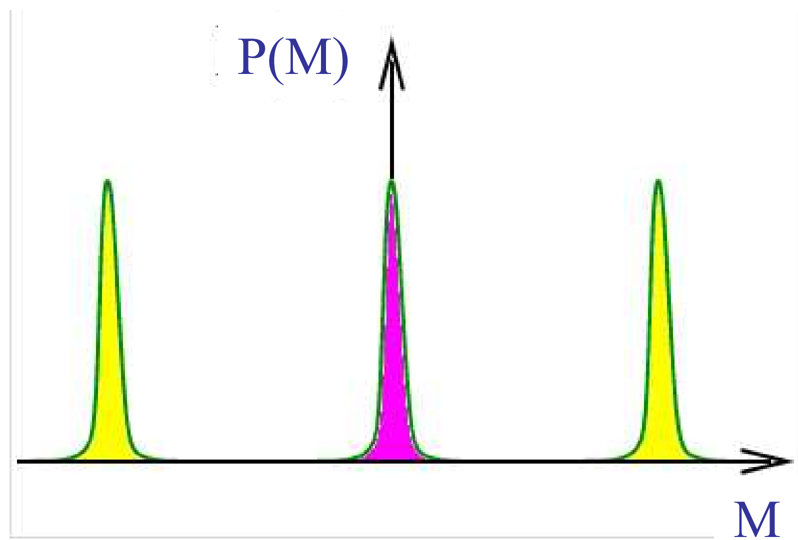
Aims: (1) Construct a picture gallery of scaling functions (PDF-s)
(2) Identify underlying nonequilibrium features by comparing PDF-s

Results: picture gallery - surface growth experiments
scalability of parallel algorithms,
turbulence and surface fluctuations,
upper critical dimension of KPZ equation
1/f noise and extreme statistics

Gaussian and nongaussian distributions I



Extensive quantity in a noncritical system $(\xi \ll L)$ — central limit theorem → Small Gaussian fluctuations around the mean



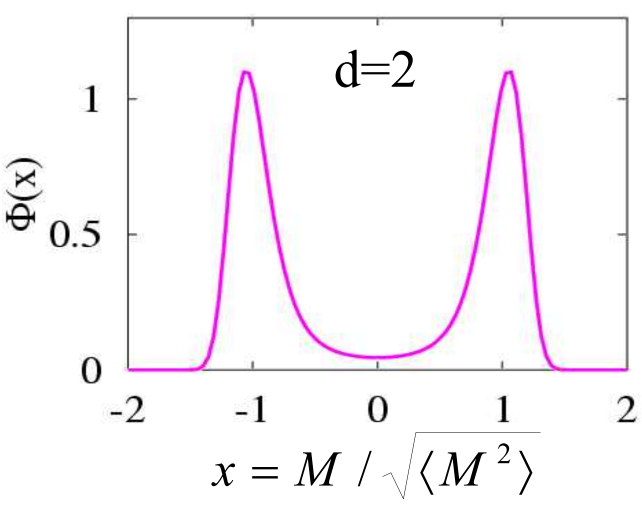
Example: Ising model above and below Tc.

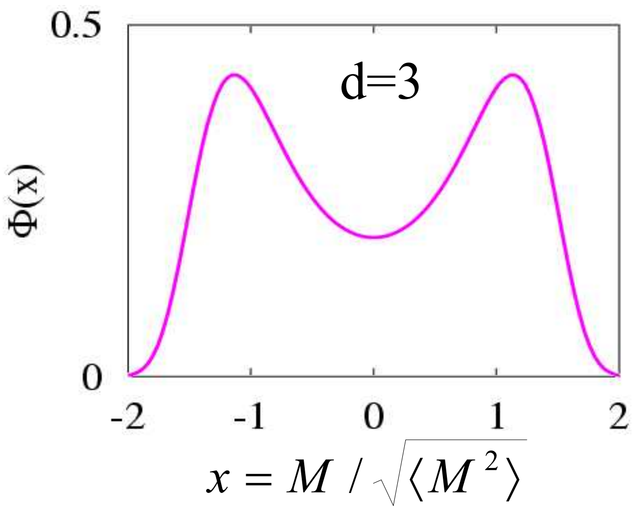
Gaussian and nongaussian distributions II

Extensive quantity in a critical system $(\xi \sim L)$ $\xrightarrow[\text{theorem}]{\text{no central limit}}$ nongaussian fluctuations around the mean

Example: Ising model at Tc - emergence of universal scaling functions

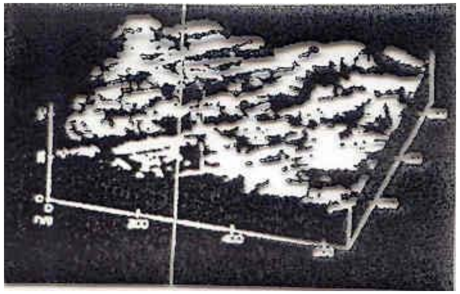
Scaling variable: $\boxed{x = M / \sqrt{\langle M^2 \rangle}}$ Scaling func.: $\boxed{\Phi(x) = \sqrt{\langle M^2 \rangle}\, P(M)}$



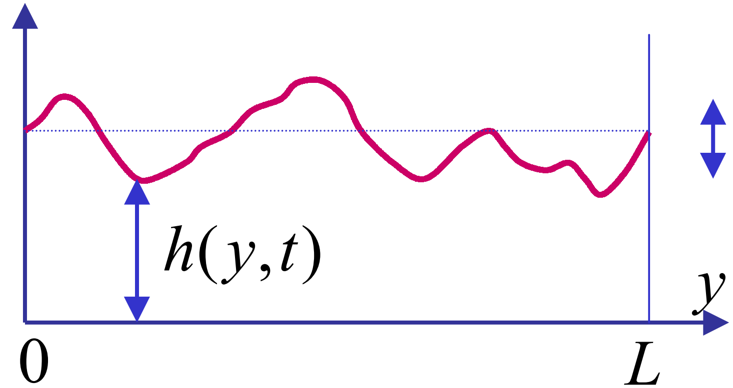


Boundary condition dependence - a characteristics of nonequilibrium systems

Nongaussianity -- width distribution of interfaces



Ag on glass



$$\sqrt{\langle w_2 \rangle} \sim L^{\chi}$$

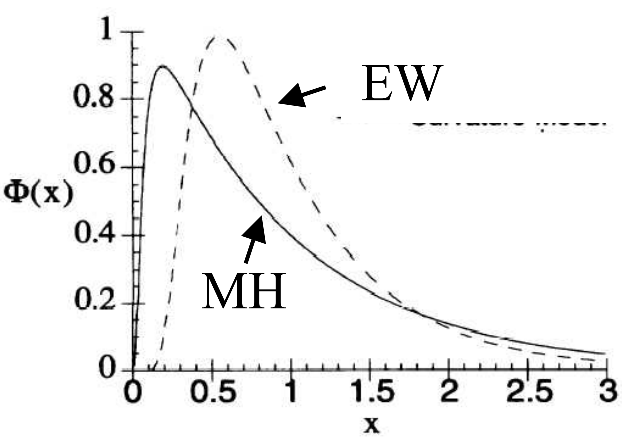
width

$$w_2(L,t) = \frac{1}{L}\int_0^L dy\left[h(y,t) - \bar{h}\right]^2$$

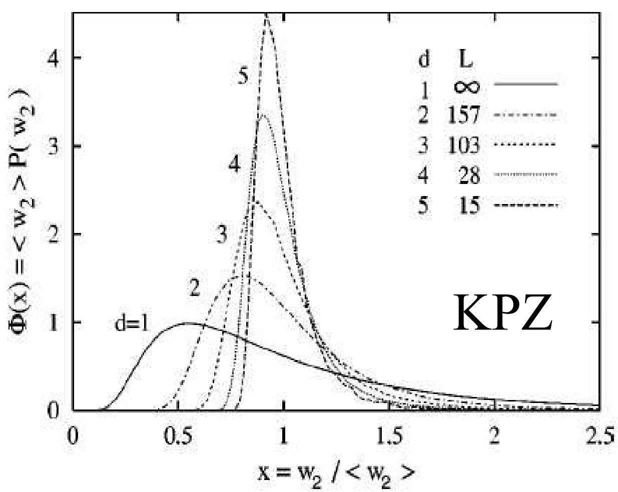
Stationary distribution:

$$\sqrt{\langle w_2 \rangle_L}\; P(w_2) = \Phi\left(\frac{w_2}{\langle w_2 \rangle_L}\right) = \Phi(x)$$

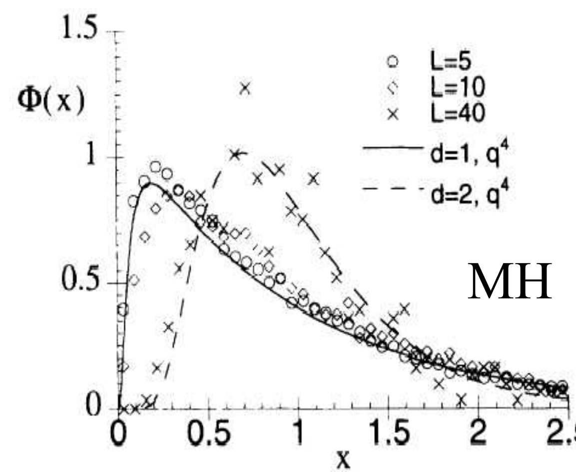
Picture gallery



PRE50, 639, 3589 (1994)



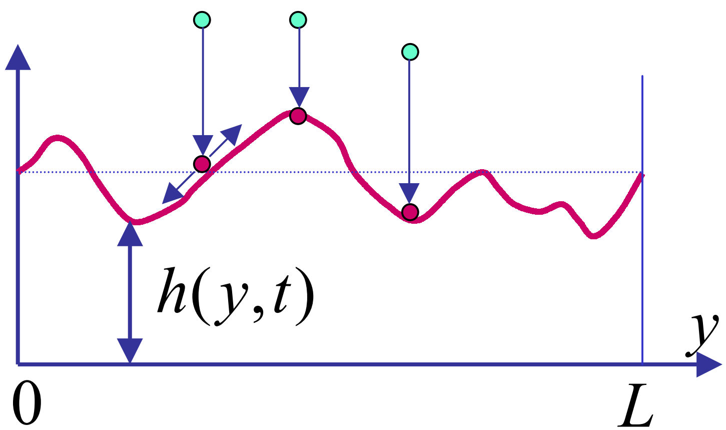
PRE65, 026136 (2002)



PRE50, 3530 (1994)

Edwards-Wilkinson (EW) interface

Surface tension driven growth



Vertical growth velocity can depend only on the spatial derivatives of h .

$$\mathrm{v} = \partial_t h = f(\partial_y h, \partial_y^2 h, \ldots) + \eta$$

Noise in the arrival (Gaussian, white is assumed)

Long wavelength (gradient) expansion:

$$f(\partial_y h, \partial_y^2 h, \ldots) = \mathrm{v}_0 + \sigma\, \partial_y^2 h + \ldots$$

average growth velocity

surface tension

In the system moving with v_0:

$$\partial_t h = \sigma \Delta h + \eta$$

Fourier modes:

$$h(y) = \sum_k e^{iky} h_k$$

Steady state distribution function:

$$\mathbf{P}[h] \sim e^{-\frac{\sigma}{2}\int_0^L (\nabla h)^2 dy}$$

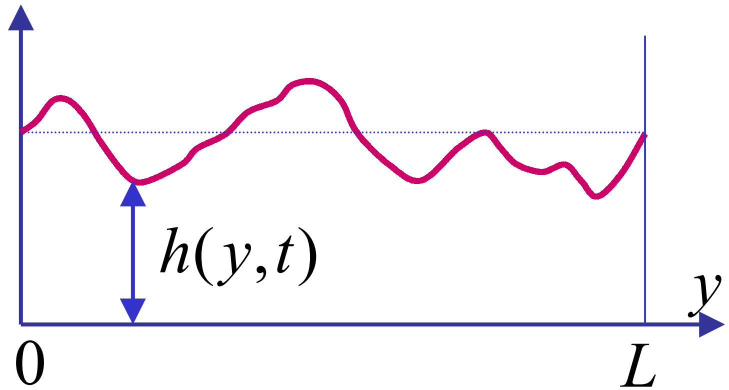
$$\mathbf{P}[h_k] \sim e^{-\frac{\sigma}{2}L\sum_k k^2 |h_k|^2} = \prod_k e^{-\frac{\sigma}{2}Lk^2|h_k|^2}$$

Independent modes

Nongaussian distributions - Edwards-Wilkinson (EW) interface

Surface tension driven dynamics

$$\boxed{\partial_t h = \sigma \Delta h + \eta}$$

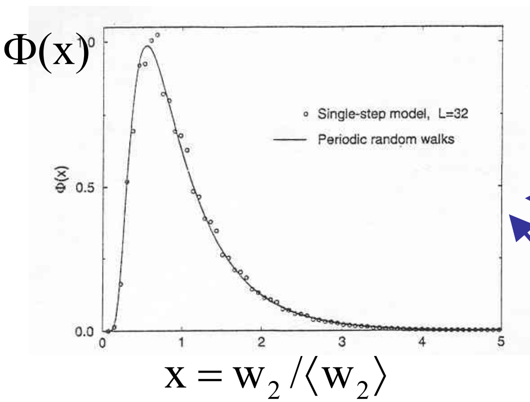


Stationary state:

$$\boxed{\mathbf{P}[h] \sim e^{-\frac{\sigma}{2}\int_0^L (\nabla h)^2 dy}}$$

Independent Fourier modes:

$$\boxed{\mathbf{P}[h_k] \sim e^{-\frac{\sigma}{2}L\sum_k k^2 |h_k|^2} = \Pi_k e^{-\frac{\sigma}{2}Lk^2 |h_k|^2}}$$



?

$$\boxed{\langle |h_k|^2 \rangle \sim \frac{1}{k^2}}$$

nonidentical, singular fluctuations

diverging fluctuations

$$\boxed{w_2 = \frac{1}{L}\int_0^L dy[h(y,t) - \bar{h}]^2 = \sum_k |h_k|^2 \propto L}$$

sum of independent variables

Reason for the failing of the central limit theorem

Derivation of the width distribution for EW interface (1)

$$\boxed{\partial_t h = \sigma \Delta h + \eta}$$

Stationary state: $$\mathbf{P}[h] \sim e^{-\frac{\sigma}{2}\int_0^L (\nabla h)^2\, dy} \quad \Rightarrow \quad \prod_k e^{-\frac{\sigma}{2} L k^2 |h_k|^2}$$

Independent Fourier modes

Width distribution:

$$P(w_2) = \int D(h)\, \mathbf{P}(h)\, \delta\left[w_2 - \frac{1}{L}\int_0^L dy (h - \overline{h})^2\right]$$

Generating function:

$$G(s) = \int_0^\infty dw_2 e^{-w_2 s} P(w_2) = \int D(h)\, e^{-\frac{\sigma}{2}\int_0^L (\nabla h)^2 dy - \frac{s}{L}\int_0^L (h - \overline{h})^2 dy}$$

Path integral of harmonic oscillator

In terms of Fourier modes:

$$= N \prod_k \int_{-\infty}^{\infty} dh_k \int_{-\infty}^{\infty} dh_{-k} e^{-(\sigma k^2 + \frac{2s}{L})|h_k|^2} = \widetilde{N} \prod_k \left[\sigma\, k^2 + \frac{2s}{L}\right]^{-1}$$

Normalization:

$$G(0) = 1 \qquad G(s) = \prod_k \frac{\sigma\, k^2}{\sigma\, k^2 + \frac{2s}{L}} \quad \Rightarrow \quad \boxed{G(s) = \prod_{n=1}^{\infty} \left[1 + \frac{Ls}{2\pi^2 \sigma\, n^2}\right]^{-1}}$$

$$k = \frac{2\pi}{L} n$$

Derivation of the width distribution for EW interface (2)

Generating function:

$$G(s)=\int_0^\infty dw_2 e^{-w_2 s}P(w_2) \;=\prod_{n=1}^{\infty}\left[1+\frac{Ls}{2\pi^2\sigma n^2}\right]^{-1}$$

Average width:

$$\langle w_2\rangle=-\frac{dG}{ds}\bigg|_{s=0}=\frac{L}{2\pi^2\sigma}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{L}{12\sigma}$$

$$\prod_{n=1}^{\infty}\left[1+\frac{6s\langle w_2\rangle}{\pi^2 n^2}\right]^{-1}=F(s\langle w_2\rangle)$$

Width distribution in scaling form:

$$\boxed{P(w_2)}=\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}G(s)e^{w_2 s}=\frac{1}{\langle w_2\rangle}\int_{-i\infty}^{i\infty}\frac{ds\langle w_2\rangle}{2\pi i}F(s\langle w_2\rangle)e^{\frac{w_2}{\langle w_2\rangle}s\langle w_2\rangle}=\boxed{\frac{1}{\langle w_2\rangle}\Phi\left(\frac{w_2}{\langle w_2\rangle}\right)}$$

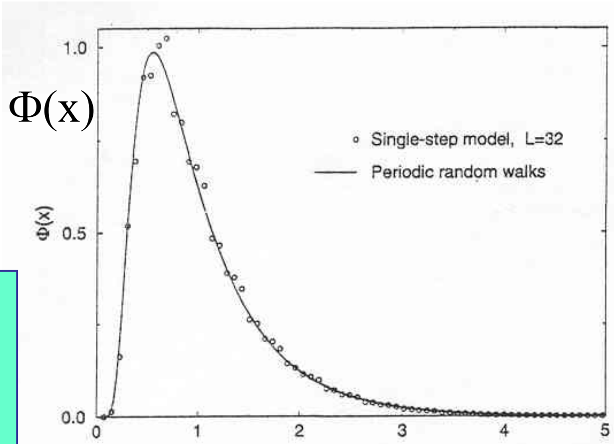
Scaling function:

$$\Phi(x)=\int_{-i\infty}^{i\infty}\frac{dy}{2\pi i}\prod_n\left(1+\frac{6y}{\pi^2 n^2}\right)^{-1}e^{xy}$$

Simple poles at
$y=-n^2\pi^2/6$

$$\boxed{\Phi(x)=\frac{\pi^2}{3}\sum_{n=1}^{\infty}(-1)^{n-1}n^2 e^{-\frac{\pi^2}{6}n^2 x}}$$

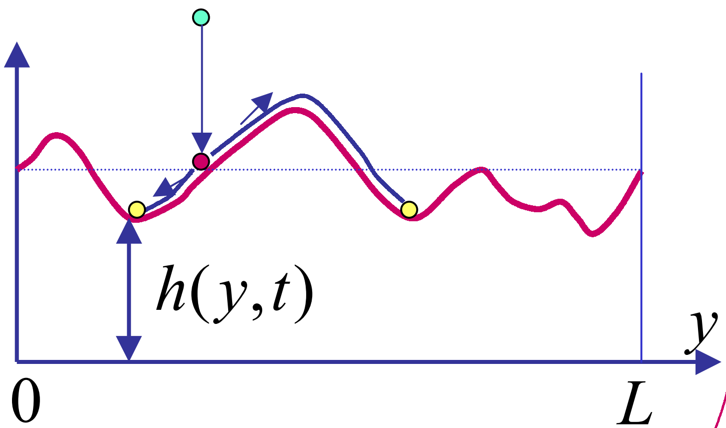
$\Phi(x)$



$x = w_2/\langle w_2\rangle$

Mullins-Herring (MH) interface

Surface diffusion driven growth:
Particles diffuse on the surface
before getting „snowed in”.



Vertical growth velocity: v_0 (average growth)
plus the flux j of particles diffusing on the surface

$$\mathrm{v} = \partial_t h = \mathrm{v}_0 - \partial_y j + \eta$$

noise in the arrival (Gaussian, white)

average growth velocity

growth due to diffusion on the surface

Flux: points towards larger curvatures.

$$j = \partial_y f(\cancel{\partial_y h}, \partial_y^2 h, \ldots) \approx D \partial_y \partial_y^2 h$$

Long wavelength expansion:

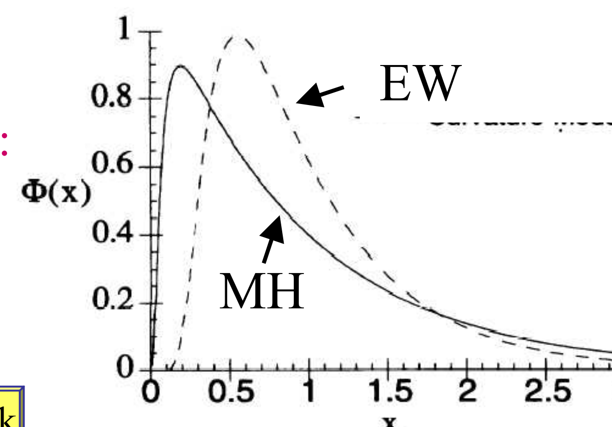
In the system moving with v_0:

$$\partial_t h = -D\Delta^2 h + \eta$$

Steady state distribution:

$$\mathbf{P}\left[h_k\right] \sim e^{-DL\sum_k k^4 |h_k|^2} = \prod_k e^{-DLk^4 |h_k|^2}$$

Comparison of width distributions:



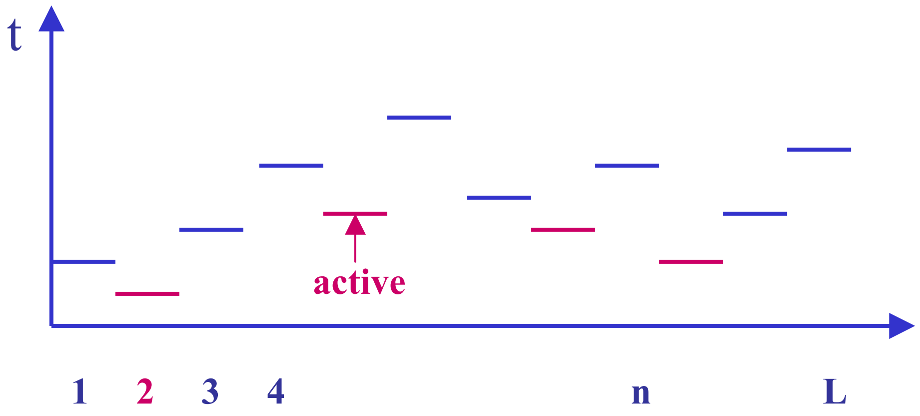
Homework

Scalability of Parallel Algorithms (fluctuating time-horizon)

G. Korniss et al. PRL84, 1351 (2000)

Local time of the computer

Only local minima can compute

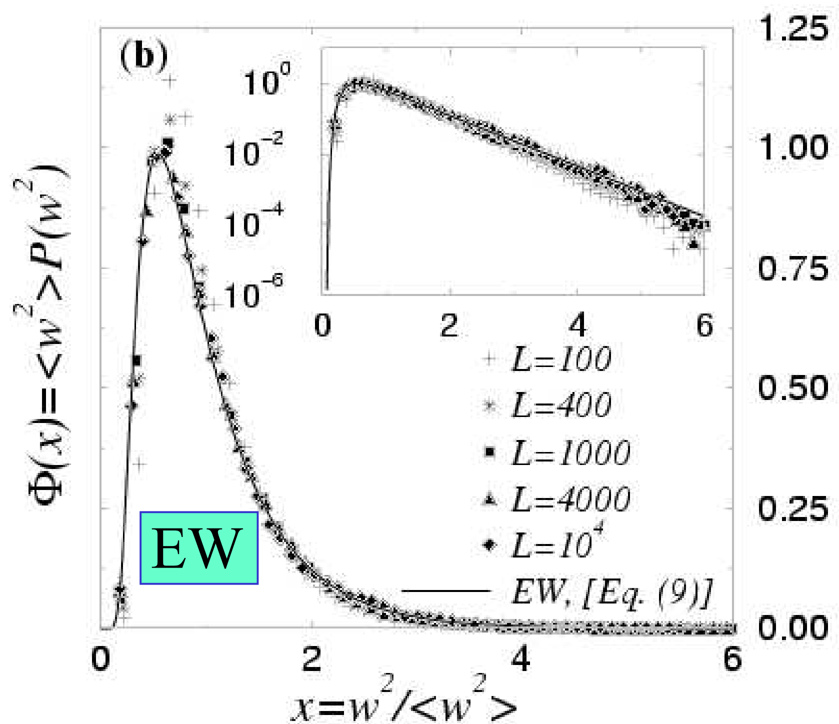


Events:
Poisson arrivals

How does the number of active computers grow with L?

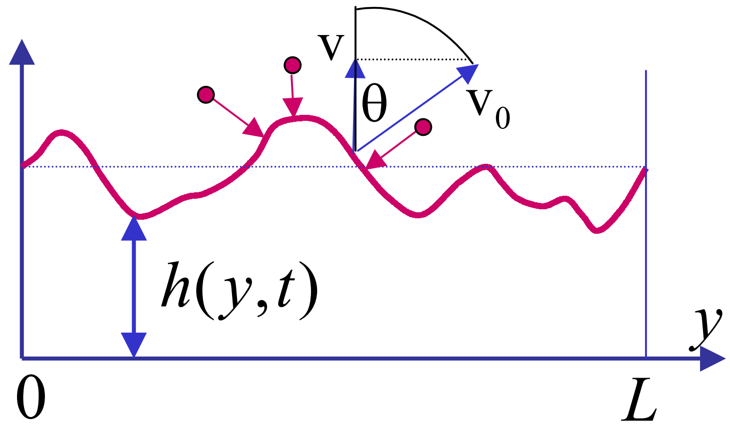
$$\boxed{N_{active} \sim L^{\alpha}}$$

$$\boxed{\alpha = 1}$$



Nongaussian distributions -- Kardar-Parisi-Zhang (KPZ) interface

PRE65, 026136 (2002)



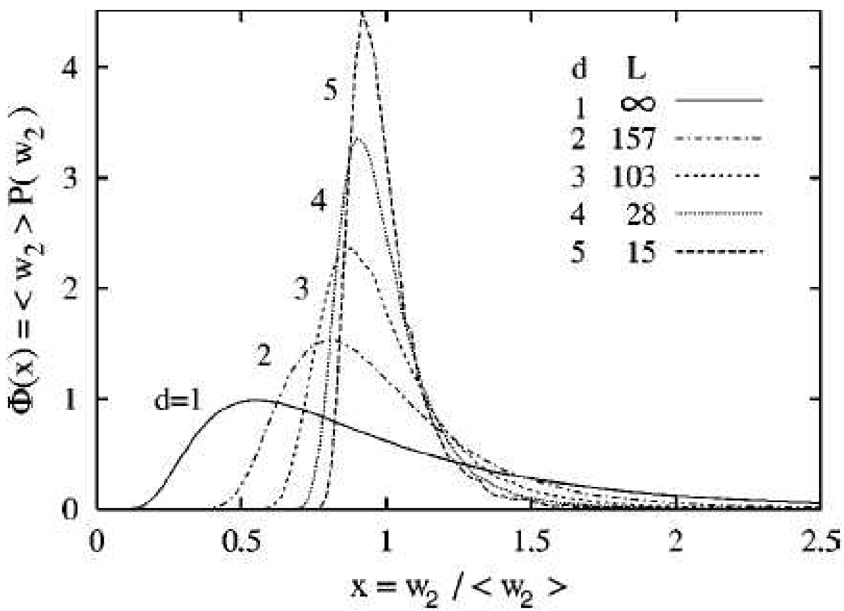
attachment: growth along the normal
+ surface tension effects

$$\partial_t h = \sigma\, \Delta h + \lambda (\nabla h)^2 + \eta$$

$$v = v_0 \cos(\theta) + \sigma\, \Delta h + \eta$$

$$\sqrt{1-\sin^2\theta} \approx \sqrt{1-(\nabla h)^2} \approx 1-(\nabla h)^2/2$$

Q: Is there an upper critical dimension?



$$\langle |h_k|^2 \rangle \sim \frac{1}{k^{2+\zeta}}$$

Nonidentical, singular,
nongaussian fluctuations

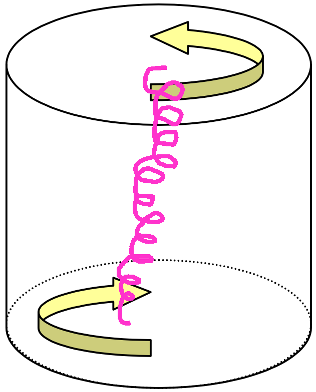
No parameters to fit

Turbulence and the d=2 EW model

S. Bramwell et al. Nature 396, 552 (1998)

Experiment:

French washing machine

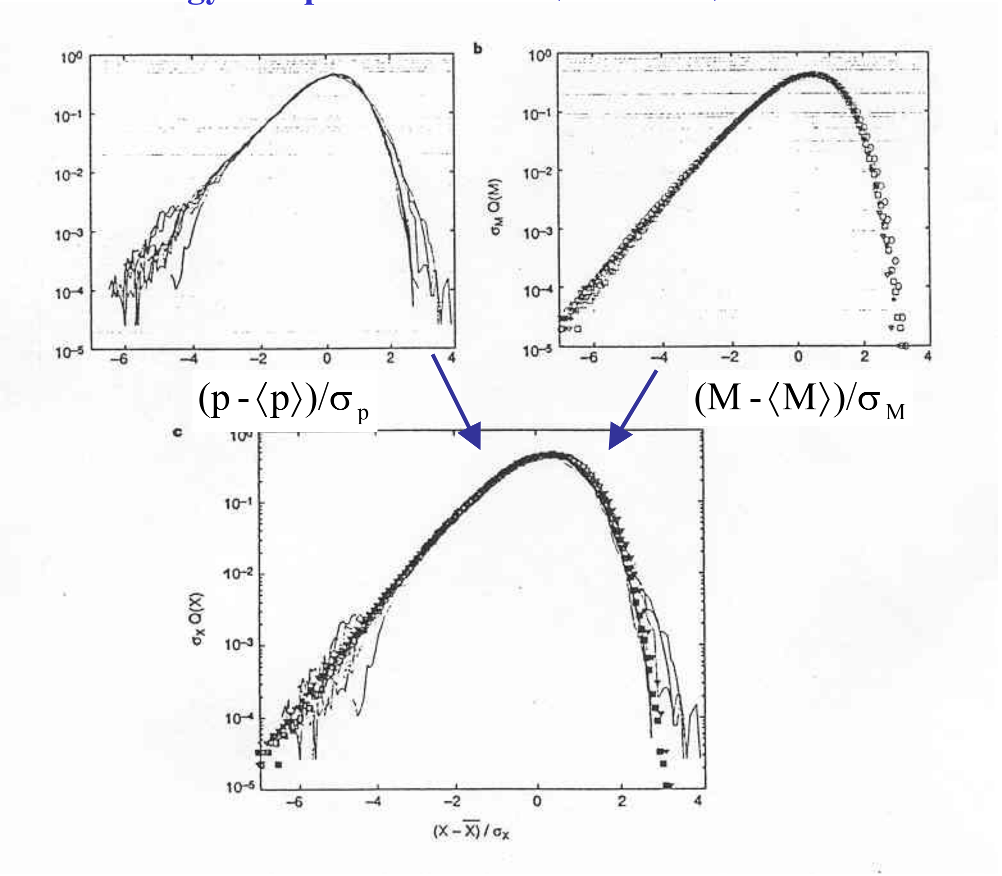


Critical

$$\boxed{\omega = \omega_c}$$

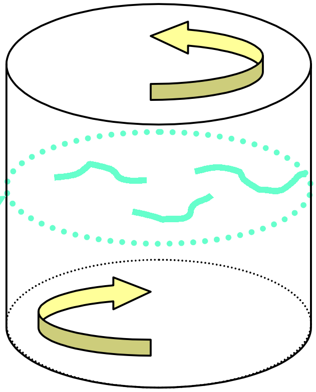
Distribution of energy dissipation

Distribution of d=2 XY magnetization below Tc. (finite-size)



Aji & Goldenfeld, PRL86, 1007 (2001)

dissipation is mainly on the fluctuations of the shear pancake

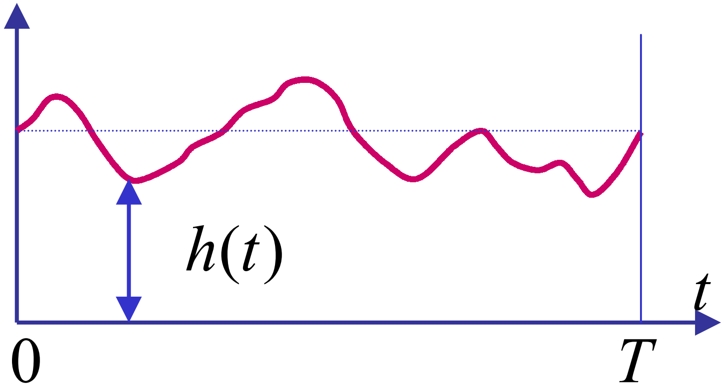


$$\boxed{d = 2 \ \ EW \equiv XY \ \ T < T_c}$$

Possible connection to extreme statistics?

Width distributions for $1/f^{\alpha}$ signals

PRL87,240601(2001)
PRE65,046140(2002)



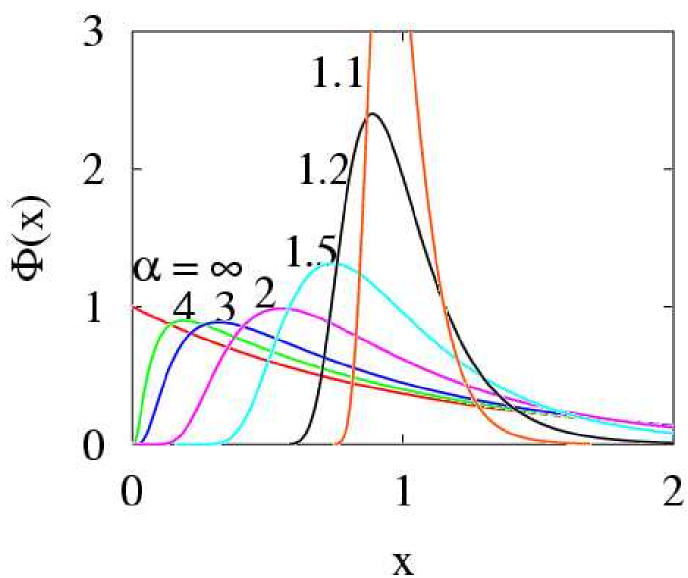
Stationary distribution for Fourier modes

$$\mathbf{P}[h_n] \sim e^{-\sigma_L \sum_n |n|^{\alpha} |h_n|^2}$$

$$w_2 = \sum_n |h_n|^2$$

Integrated power spectrum

Question: Is there an α for which extreme statistics distribution emerges?

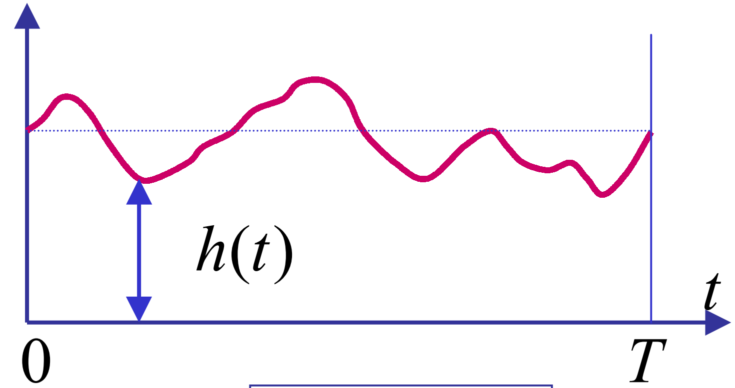


$$\sqrt{\langle w_2 \rangle_T}\; P(w_2) = \Phi_{\alpha}\left(x = \frac{w_2}{\langle w_2 \rangle_T} \right)$$

$\alpha \to 1$ $\qquad$ $\Phi_{\alpha}(x) \to \delta(x)$

Does it have and internal structure?

Extreme statistics for $\boxed{\alpha = 1}$.



$\alpha \to 1$ new scaling variable

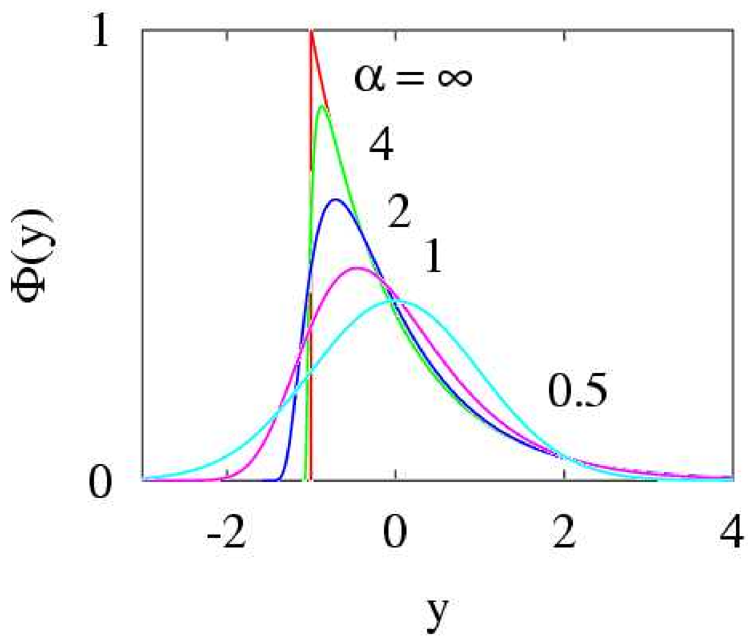
$$y = \frac{w_2 - \langle w_2 \rangle_T}{\sigma_T}$$

$$\sigma_T = \sqrt{\langle w_2^2 \rangle_T - \langle w_2 \rangle_T^2}$$

$$w_2 = \sum_n |h_n|^2$$

integrated power spectrum

$$\sigma_T P(w_2) = \Phi_\alpha(y)$$



$\alpha = 1$

$$\Phi_1(y) = e^{-y-e^{-y}}$$

Fisher-Tippett-Gumbel extreme value distribution

$\alpha \leq 1/2$

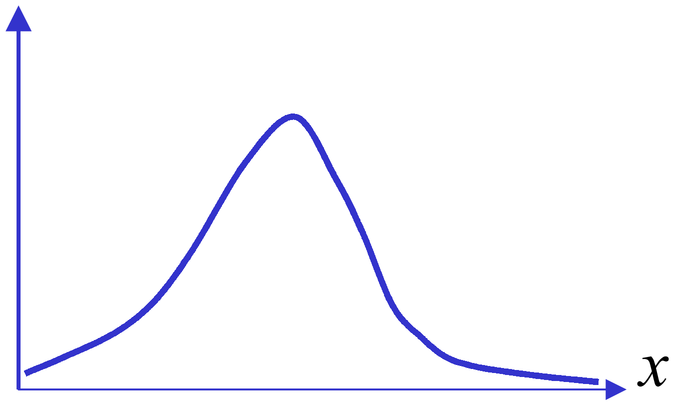
$$\Phi_\alpha(y) \sim e^{-y^2}$$

$$\langle |h_n|^2 \rangle \sim n^{-\alpha}$$

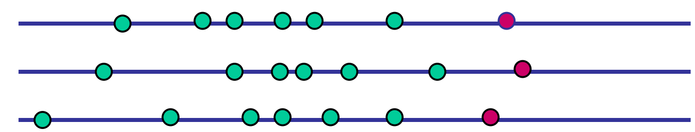
Central limit theorem is restored for $\alpha \leq 1/2$

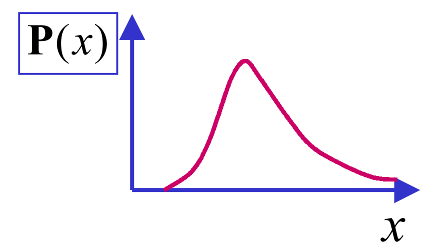
Extreme value statistics

N numbers are drawn from a distribution $P_0(x)$



Question: What is the limiting $(N \to \infty)$ distribution of the largest number?





Three types depending on $P_0(x \to \infty) = ?$

$$\rho(x) = \int_{-\infty}^{x} dy\, P(y)$$

I. Faster than any power law (Fisher -Tippet - Gumbel):

$$\boxed{\rho(x) = e^{-e^{-x}}}$$

II. Power law (Fisher-Tippett-Frechet)

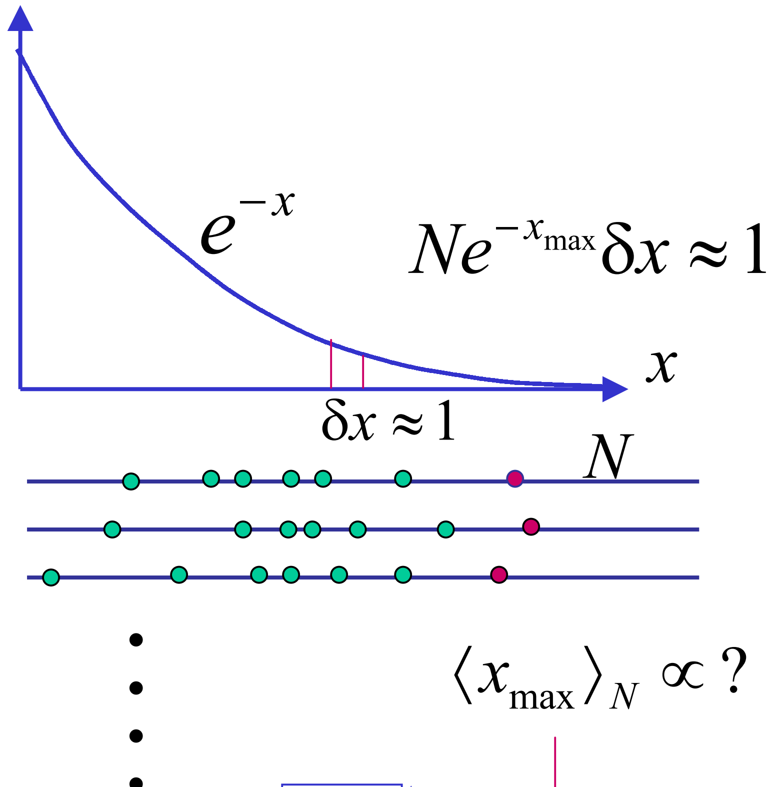
$$\boxed{\rho(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x^{-\alpha}} & x > 0 \end{cases}}$$

III. Finite cutoff (Weibull)

$$\boxed{\rho(x) = \begin{cases} e^{-(-x)^{\alpha}} & x \leq 0 \\ 1 & x > 0 \end{cases}}$$

Calculation of extreme value statistics for independent events

Simple example: Parent distribution: $P_0(x)$



Probability that all N measurements give smaller than y:

$$\rho(y) = \left[\int_0^y dy\, P_0(y)\right]^N = (1-e^{-y})^N$$

$$= \left(1 - \frac{e^{-(y-\ln N)}}{N}\right)^N$$

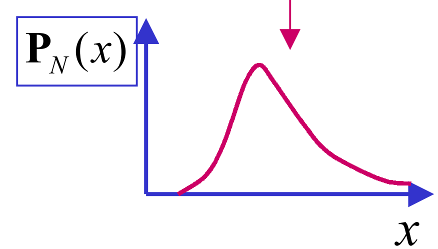
Moving frame

$z = y - \ln N$

$$\boxed{\rho(z) = e^{-e^{-z}}}$$

Fisher -Tippet - Gumbel: $$\boxed{P(z) = e^{-z-e^{-z}}}$$

$\langle x_{\max}\rangle_N \propto ?$

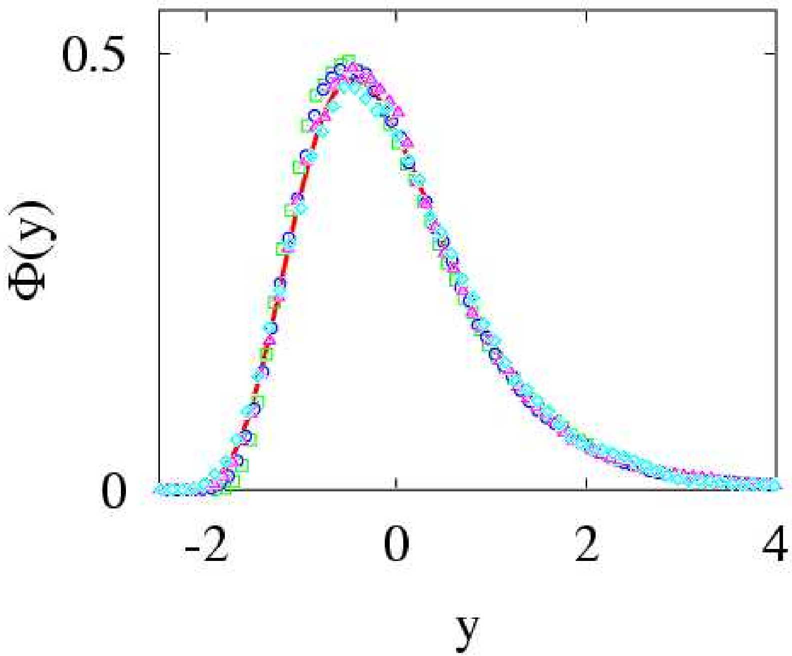


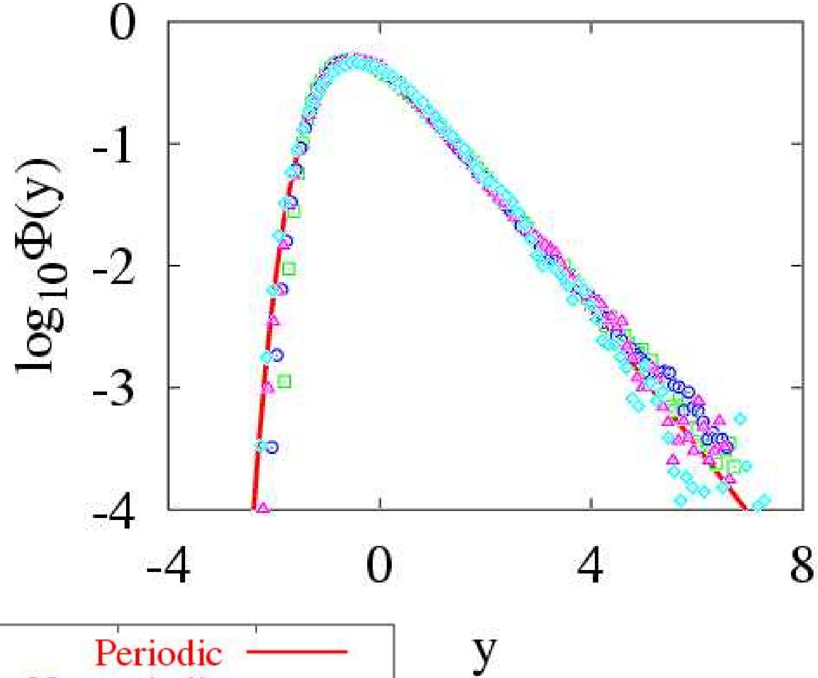
$x_{\max} \approx \ln N$

Homework

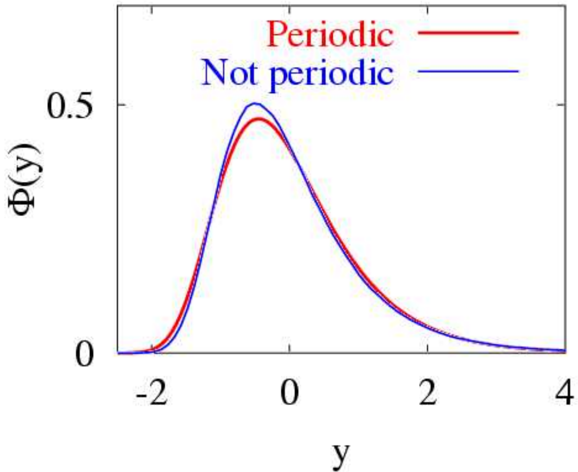
Extreme value statistics - resistivity fluctuations

Experimental data on GaAs films (A.V. Yakimov and F.N. Hooge)



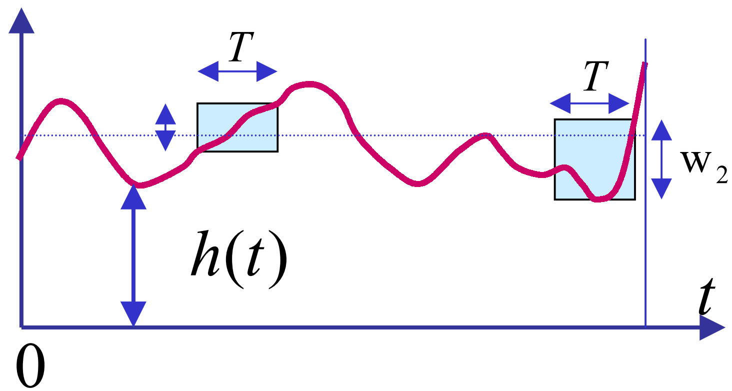


Problem: Scaling functions depend on boundary conditions



Boundary condition effects

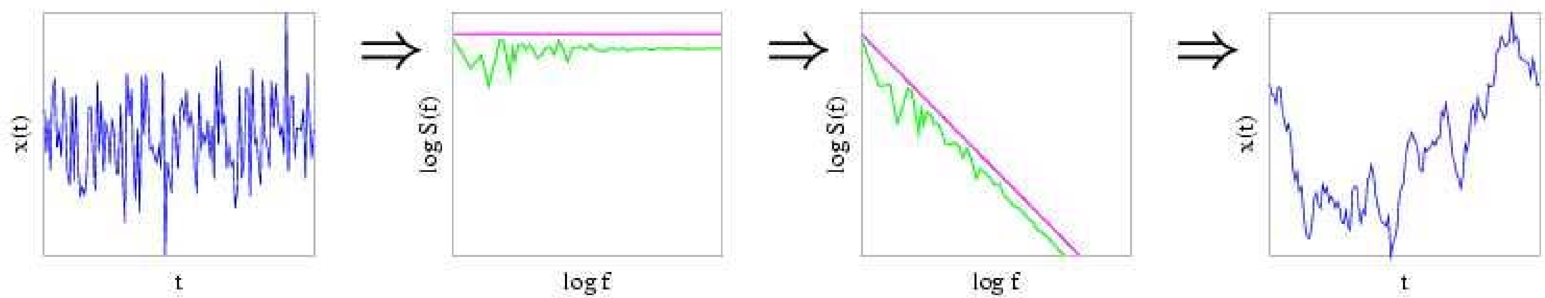
Nonperiodic signals



$\Longrightarrow \quad P(w_2)$

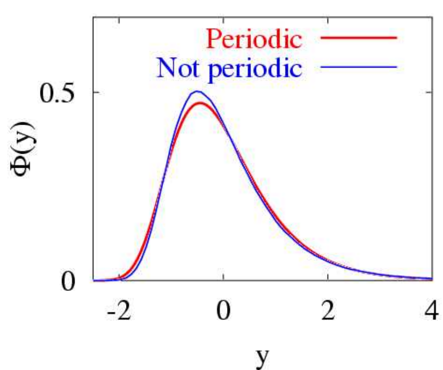
Exact results for $\alpha = 0.5, 2, \infty$

Numerics:



Window b.c.:

Small difference for $\alpha = 1$



The difference increases with increasing α.

Example with $\alpha = 2$ $\Longrightarrow$

BC effects: probability distributions of the d=1 Ising model

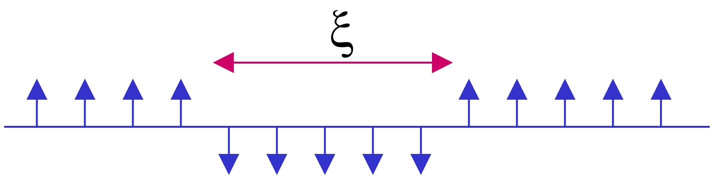
cond-mat/0308442

Ising model

$$\boxed{H = -J\sum_n \sigma_n \sigma_{n+1}}$$

Finite-size scaling limit

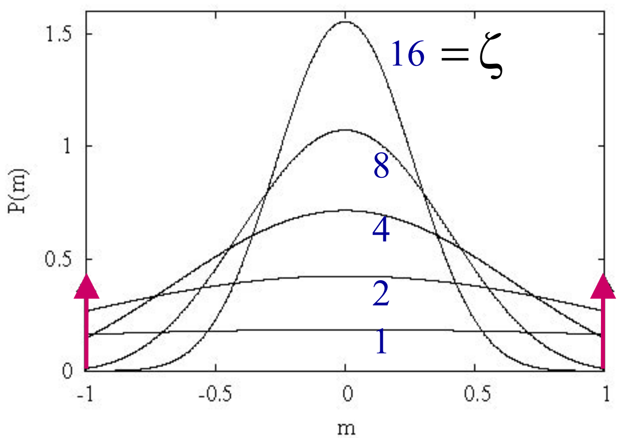
$$L \to \infty \qquad \xi \propto [1-\tanh(J/T)]^{-1} \to \infty$$



number of domain walls $\boxed{\dfrac{L}{\xi} = 2\zeta}$ fixed

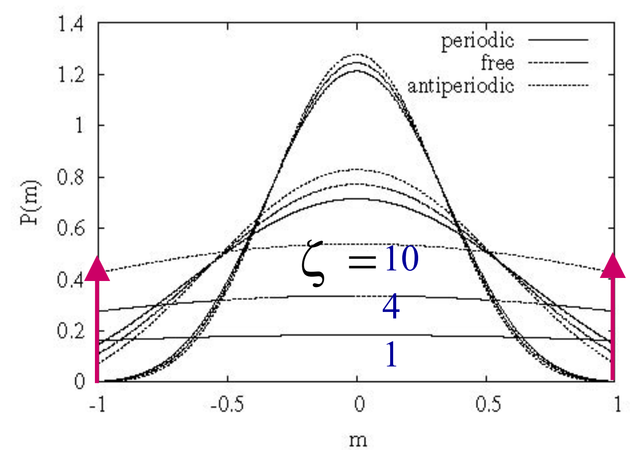
Periodic BC:

$$\boxed{P(m) = \frac{1}{2\cosh(\zeta)}\left[\delta(m+1) + \delta(m-1) + \frac{\zeta}{\sqrt{1-m^2}} I_1(\zeta\sqrt{1-m^2})\right]}$$



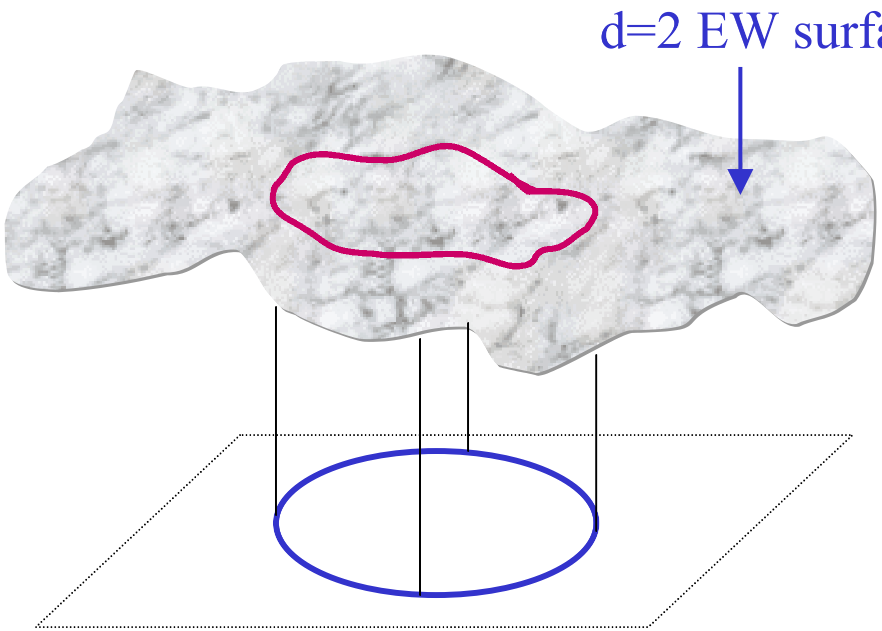
$\cancel{\delta(m\pm1)}$ for antiperiodic BC

BC dependence:



Periodic 1/f signal

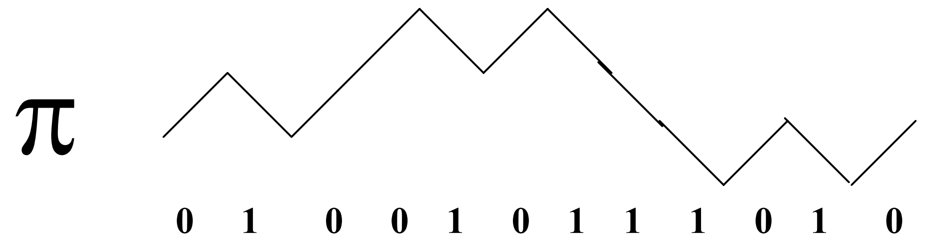
Are there periodic physical systems displaying 1/f fluctuations?



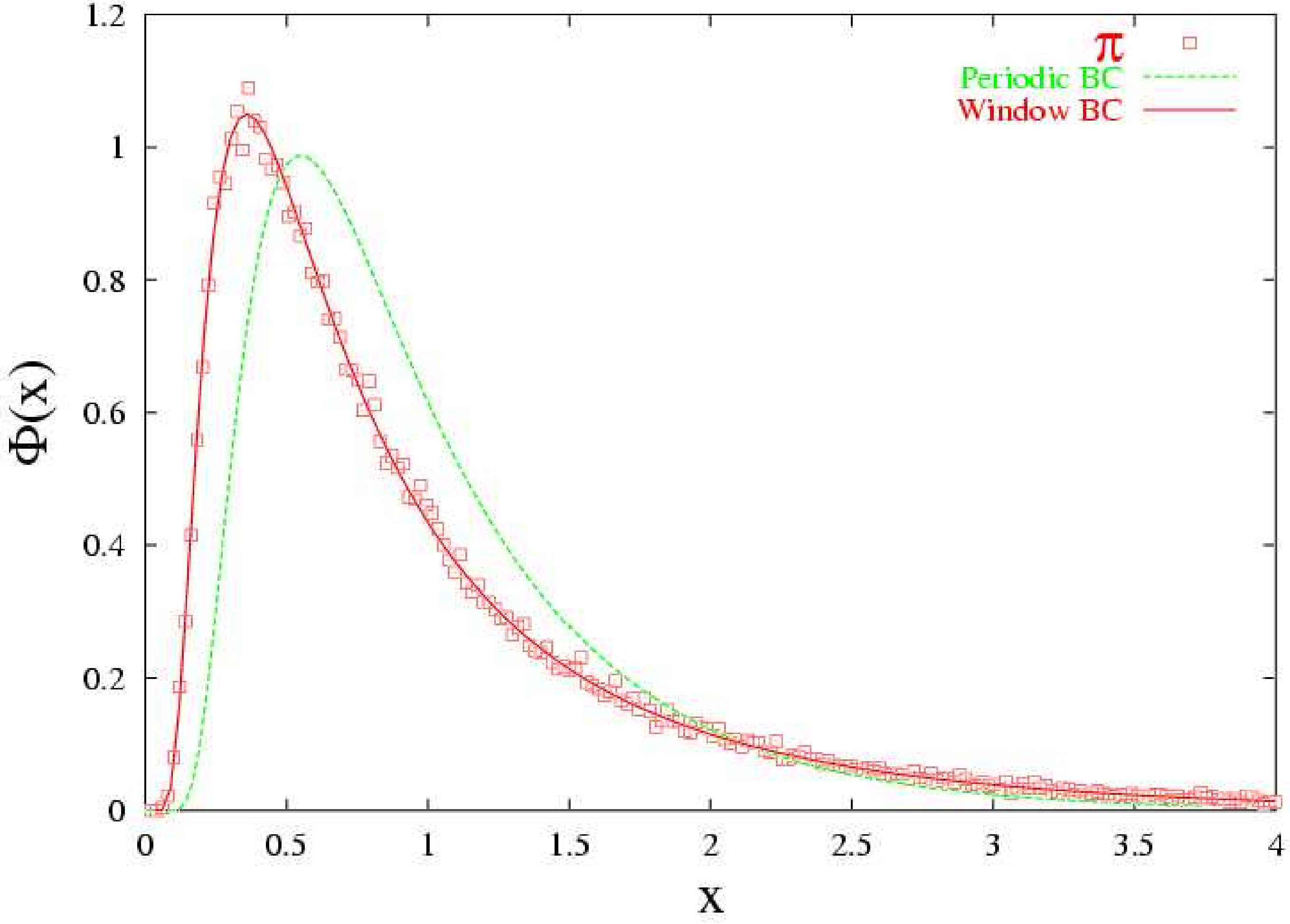
The fluctuations of red line display a perfect 1/f power spectrum.

Randomness of the digits of π

Mapping of π onto a surface



Roughness-
distribution
of π



Observing the
right boundary
conditions is
clearly important.

Homework

(1) Calculate the width distribution for the Mullins-Herring model.

(2) Find the extreme value distribution independent trials if the parent distribution is of power-law form:

$$P_0(x) = \frac{x^{-\mu}}{\mu - 1} \qquad 1 \le x \le \infty \qquad 1 \le \mu$$