Patterns Formation: Appendix I

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Patterns from stability analysis (simple examples)

(1) Local and global approaches.

Problem of relative stability in far from equilibrium systems.

(2) Linear stability analysis.

Stationary (fixed) points of differential equations.

Behavior of solutions near fixed points: stability matrix and eigenvalues.

Example: Two dimensional phase space structures

Lotka-Volterra equations, story of tuberculosis

Breaking of time-translational symmetry: hard-mode instabilities

Example: Hopf bifurcation: Van der Pole oscillator

Soft-mode instabilities: Emergence of spatial structures

Example: Chemical reactions - Brusselator.

Literature

M. C. Cross and P. C. Hohenberg, **Pattern Formation Outside of Equilibrium**, Rev. Mod. Phys. 65, 851 (1993).

J. D. Murray, **Mathematical Biology**, (Springer, 1993; ISBN-0387-57204).

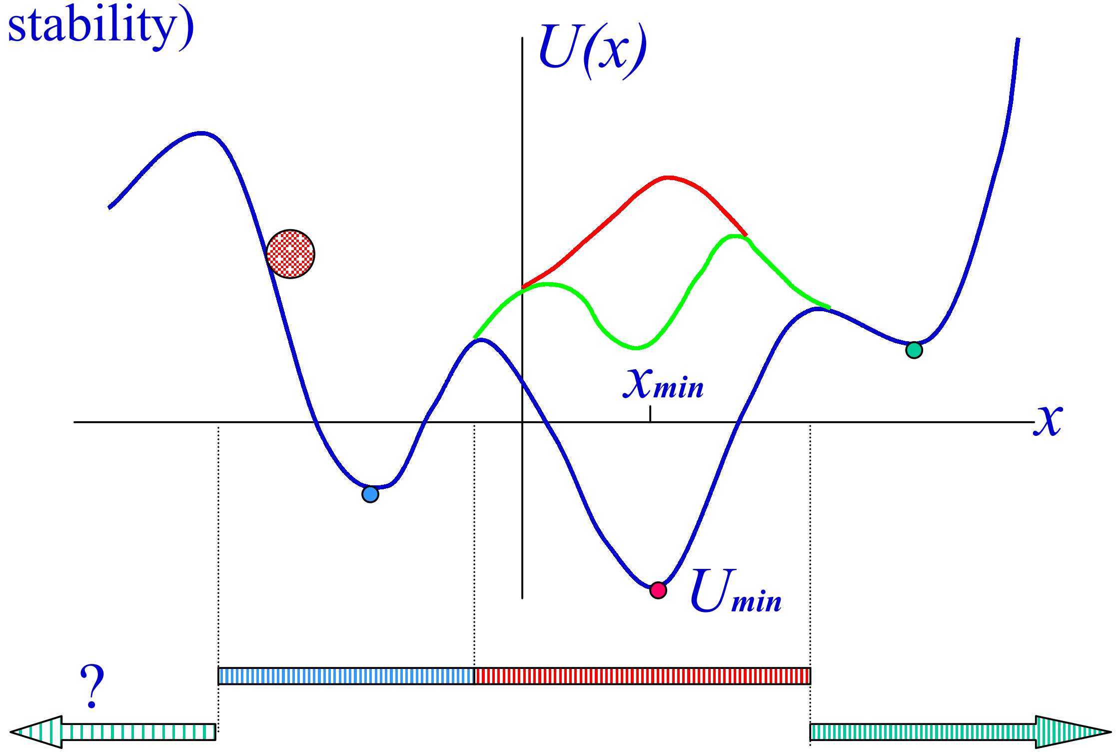
Local and global approaches

Stability (absolute and relative stability)

$$\ddot{x} = -\frac{dU}{dx}$$

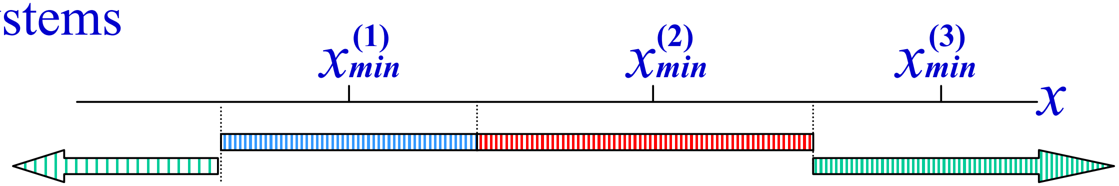
Stationary points

$$\frac{dU}{dx} = 0$$



Regions of attraction of stationary points

The problem of non-potential systems



Linear stability analysis

$$\dot{n}_1 = f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = f_2(n_1, n_2, \lambda)$$

Assumption: The system is desribed by autonomous differential equations

fields   control parameter

Stationary (fixed) points of the equations: n_1^*, n_2^*

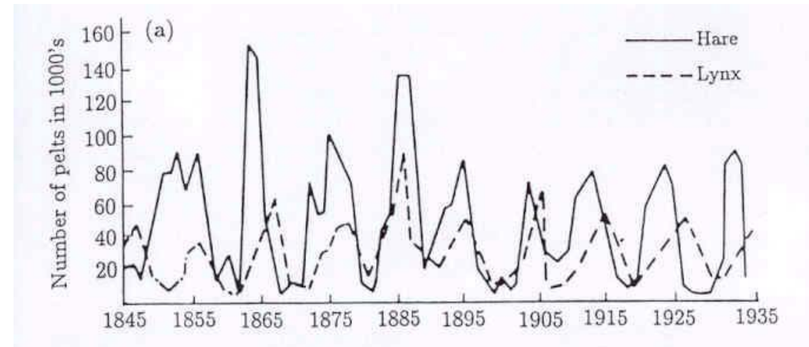
$$f_1(n_1^*, n_2^*, \lambda) = 0$$
$$f_2(n_1^*, n_2^*, \lambda) = 0$$

Lotka-Volterra systems: Hare-lynx problem

Lotka 1925 (osc. chem. react.)
Volterra 1926 (fish pop.)

n_1 - concentration of hare (rabbits)

n_2 - conc. of lynx (foxes)



αn_1 - rabbits eat grass and multiply

($\alpha = 1$ sets the time-scale)

$-\lambda n_2$ - foxes perish without eating

$-\beta n_1 n_2$ - rabbits perish when meeting foxes ($\beta = 1$ sets the n_1 -scale)

$-\gamma n_1 n_2$ - foxes multiply when meeting rabbits ($\gamma = 1$ sets the n_2-scale)

$$\dot{n}_1 = n_1 - n_1 n_2$$
$$\dot{n}_2 = n_1 n_2 - \lambda n_2$$

$-\kappa n_1^2$ - finite grass supply

Fixed points: $n_1^* = 0, \; n_2^* = 0$ $\quad n_1^* = \lambda, \; n_2^* = 1$

Hare-lynx problem: Fixed point structure

$$\dot{n}_1 = n_1 - n_1 n_2$$
$$\dot{n}_2 = n_1 n_2 - \lambda n_2$$

$n_1^* = 0, \; n_2^* = 0$

Doom

$$\mathbf{A}_\lambda \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}$$

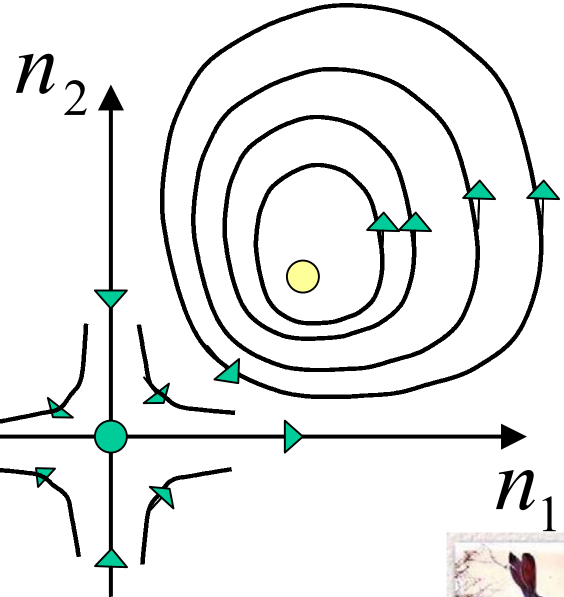
$$\omega_1 = 1 \quad \omega_2 = -\lambda$$

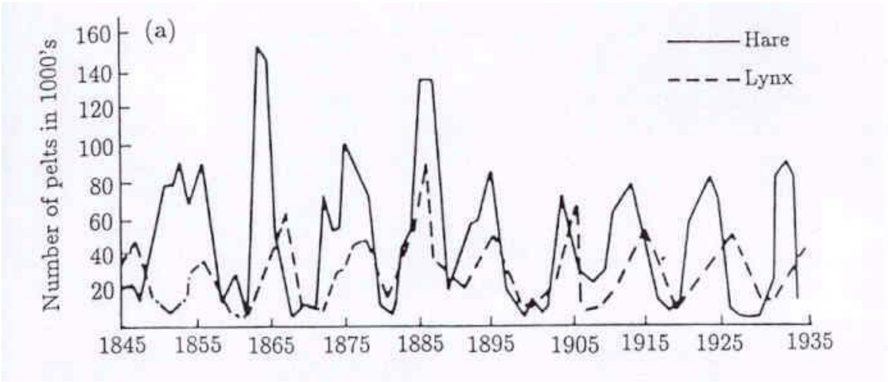
$n_1^* = \lambda, \; n_2^* = 1$

Coexistence

$$\begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix}$$

$$\omega_{1,2} = \pm i\sqrt{\lambda}$$





Problems of fluctuations and discreteness

Behavior of solutions near fixed points

Linearization

$$\dot{n}_1 = f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = f_2(n_1, n_2, \lambda)$$

$$n_1^*, n_2^*$$

$$n_1 = n_1^* + \delta n_1(t)$$
$$n_2 = n_2^* + \delta n_2(t)$$

$$\begin{pmatrix} \delta \dot{n}_1 \\ \delta \dot{n}_2 \end{pmatrix} = \mathbf{A}_\lambda \begin{pmatrix} \delta n_1 \\ \delta n_2 \end{pmatrix}$$

Stability matrix

Diagonalization

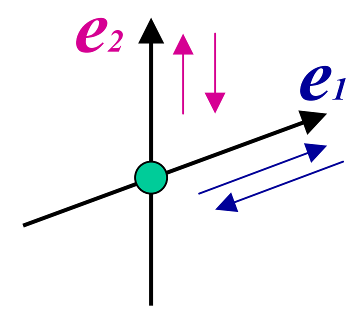
$$\tilde{\mathbf{A}}_\lambda = \begin{pmatrix} \omega_{1\lambda} & 0 \\ 0 & \omega_{2\lambda} \end{pmatrix}$$

Eigenvalues

Solution

$$\begin{pmatrix} \delta n_1 \\ \delta n_2 \end{pmatrix} = c_1 \mathbf{e_1} \, e^{\omega_{1\lambda} t} + c_2 \mathbf{e}_2 \, e^{\omega_{2\lambda} t}$$

Eigenvectors



Fixed point structures

$$\dot{n}_1 = f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = f_2(n_1, n_2, \lambda)$$

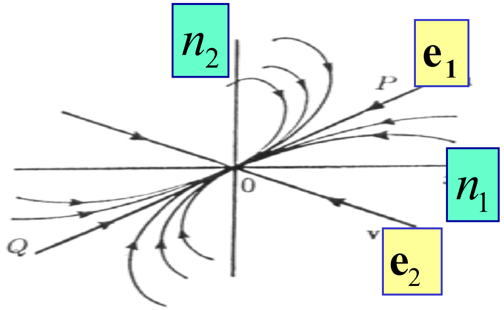
$\rightarrow$ $\omega_1, \omega_2, \mathbf{e}_1, \mathbf{e}_2$ $\rightarrow$

Re Im

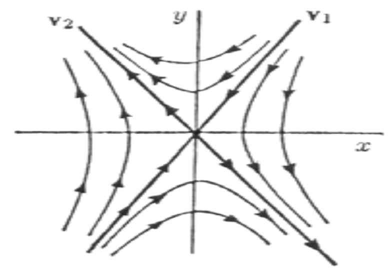
$\omega \rightarrow +, 0, - \quad \pm i$

Re $- < -$

Node I



(a)



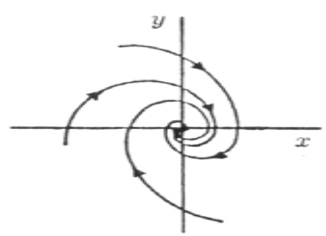
(b)

Re $+ \; -$

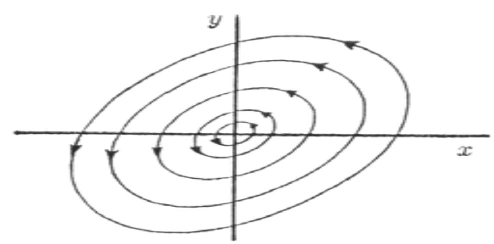
Saddle

$-$ Im

Spiral



(c)



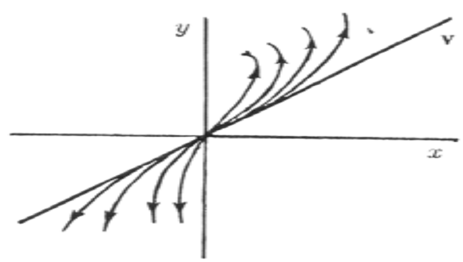
(d)

0 Im

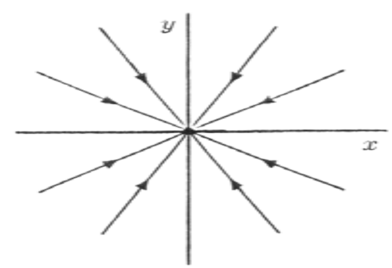
Centre

Re $+ \; = +$

Node II



(e)



(f)

Re $- = -$

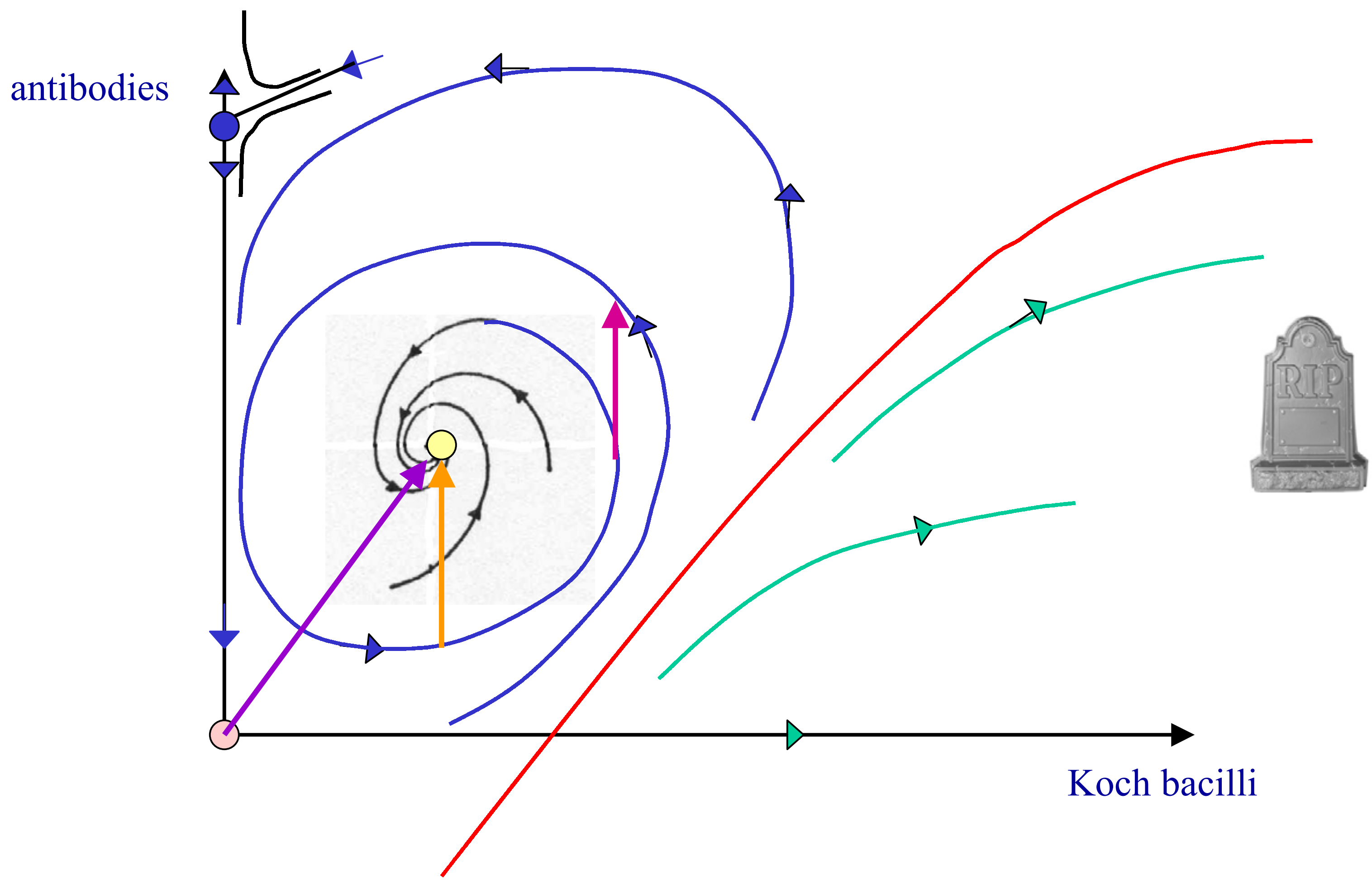
Star

$e^{\omega t}, \quad te^{\omega t}$

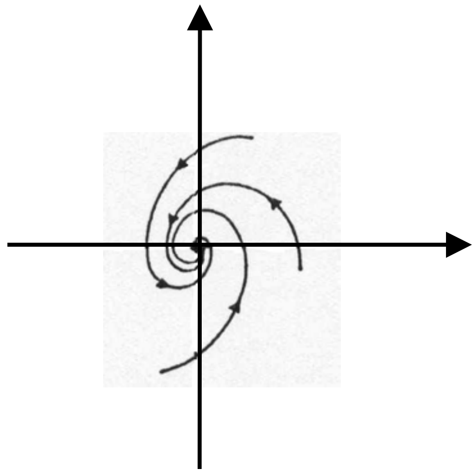
Fixed point structures and the problem of tuberculosis

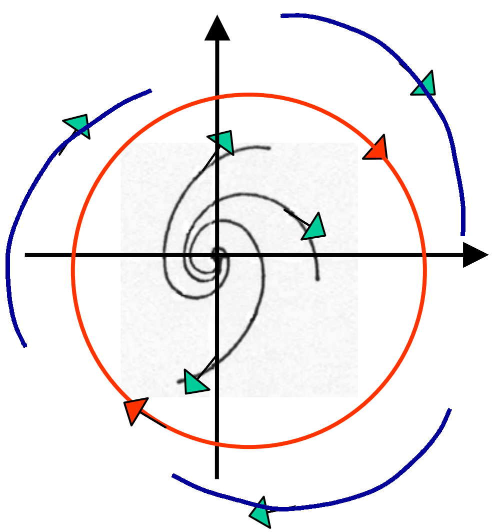
Cause: Koch bacillus; Treatment: serum, antibiotics; Immunity by vaccination

Characteristics: periodicity in the course of illness

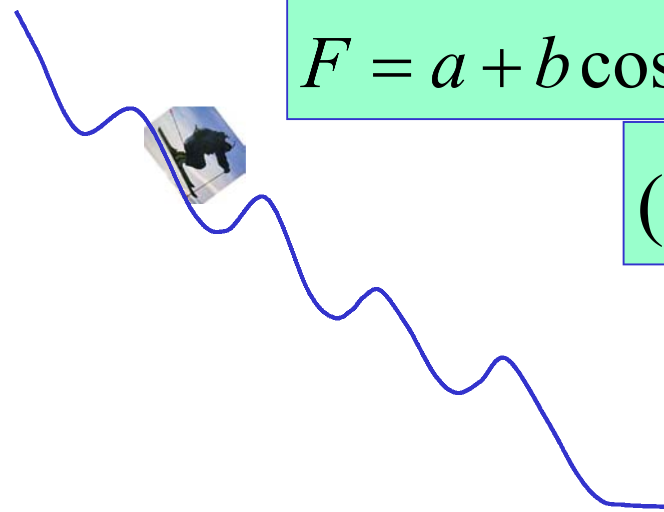


Breaking of time-translational symmetry: Limit cycles





Example: Skier on a wavy slope



$$F = a + b\cos(x) - c\,\dot{x}$$

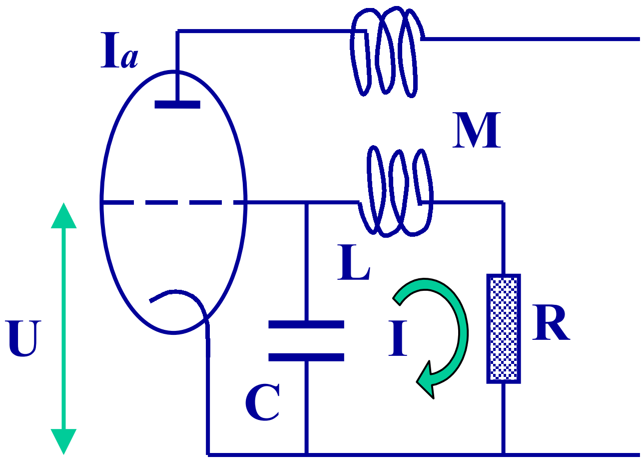
$$(x, \dot{x})$$

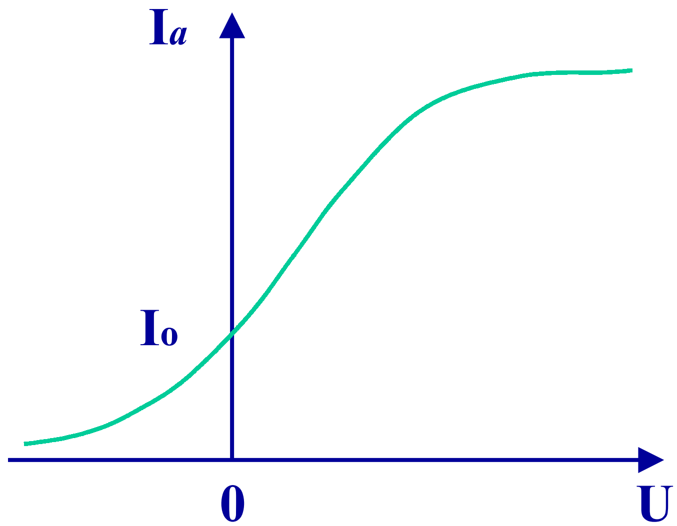
Example: Van der Pole oscillator

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x$$

$$(\dot{x} \sim I, x \sim U)$$

Van der Pole oscillator





$$I_a = I_0 + sU - \frac{g}{3}U^3$$

$$L\frac{dI}{dt} + M\frac{dI_a}{dt} + IR + \frac{1}{C}\int_0^t I d\tau = 0$$

$$x = \sqrt{\frac{Mg}{Ms - RC}}U$$

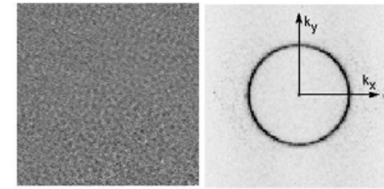
$$I = C\sqrt{\mu}\dot{x}$$

$$t \to \sqrt{LC}t'$$

$$\mu = \frac{Ms - RC}{\sqrt{LC}}$$

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x$$

Emergence of spatial structures: Soft mode instabilities



$$n \rightarrow n(t) \rightarrow n(x,t)$$

$$\dot{n} = f(n, \partial_x n, \partial_x^2 n, \ldots, \lambda)$$

Spatial mixing: convection, diffusion, … → Reaction-diffusion systems

$$\dot{n}_1 = D_1 \Delta n_1 + f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = D_2 \Delta n_2 + f_2(n_1, n_2, \lambda)$$

Chemical reactions in gels
(model example: Brussellator)

Stability analysis:

(1) Stationary homogeneous solutions:

$$f_1(n_1^*, n_2^*, \lambda) = 0$$
$$f_2(n_1^*, n_2^*, \lambda) = 0$$

Stab

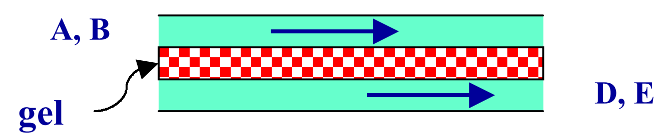
Brusselator - oscillations and spatial patterns in chemical reactions

(Prigogin and Lefever, 1968)

$$A \xrightarrow{k_1} U \qquad B + U \xrightarrow{k_2} V + D$$

$$2U + V \xrightarrow{k_3} 3U \qquad U \xrightarrow{k_4} E$$

Concentrations: **A, B, U(x,t), V(x,t)**



$$\dot{U} = D_u \Delta U + k_1 A - k_2 BU + k_3 U^2 V - k_4 U$$

$$\dot{V} = D_v \Delta V + k_2 BU - k_3 U^2 V$$

Brusselator I - rescalings and canonical form of the equations

A → U

B + U → V + D

2U + V → 3U

U → E

$$\dot{U} = D_u \Delta U + k_1 A - k_2 B U + k_3 U^2 V - k_4 U$$
$$\dot{V} = D_v \Delta V + k_2 B U - k_3 U^2 V$$

time

$$t = t' / k_4$$

space

$$x = x' \sqrt{D_u / k_4}$$

concentrations

$$U = cu \qquad V = cv \qquad c = A k_1 / k_4$$

parameters

$$d = D_v / D_u$$

$$b = k_2 B / k_4$$

equations

$$\dot{u} = \Delta u + 1 - (1+b)u + \lambda u^2 v$$
$$\dot{v} = d \Delta v + bu - \lambda u^2 v$$

control parameter

$$\lambda = k_1^2 k_3 A^2 / k_4^3$$

Brusselator II - rescalings and canonical form of the equations

A → U **B + U → V + D**

2U + V → 3U **U → E**

$$\dot{U} = D_u \Delta U + k_1 A - k_2 B U + k_3 U^2 V - k_4 U$$
$$\dot{V} = D_v \Delta V \qquad + k_2 B U - k_3 U^2 V$$

time

$$t = t' \tau$$

space

$$x = x' \ell$$

concentrations

$$U = c u \qquad V = c' v$$

equations

$$\frac{c}{\tau} \dot{u} = \frac{D_u c}{\ell^2} \Delta u + k_1 A - k_2 B c u + k_3 c^2 c' u^2 v - k_4 c u$$
$$\frac{c'}{\tau} \dot{v} = \frac{D_v c'}{\ell^2} \Delta v \qquad + k_2 B c u - k_3 c^2 c' u^2 v$$

Brusselator III - rescalings and canonical form of the equations

A → U   B + U → V + D

2U + V → 3U   U → E

$$\dot{U} = D_u \Delta U + k_1 A - k_2 BU + k_3 U^2 V - k_4 U$$
$$\dot{V} = D_v \Delta V \qquad + k_2 BU - k_3 U^2 V$$

time $t = t'\tau$ space $x = x'\ell$ concentrations $U = cu \quad V = c'v$

$\ell^2 \tau = D_u$; 1 ; 1 ; $b = k_2 B / k_4$; λ ; 1 ; $\tau = 1/k_4$

$$\dot{u} = \frac{D_u \tau}{\ell^2} \Delta u + \frac{k_1 A \tau}{c} - k_2 B \tau u + k_3 c c' \tau u^2 v - k_4 \tau u$$
$$\dot{v} = \frac{D_v \tau}{\ell^2} \Delta v \qquad + \frac{k_2 B c \tau}{c'} u - k_3 c^2 \tau u^2 v$$

$d = D_v / D_u$; $c = c' = \frac{k_1 A}{k_4}$; b ; $\lambda = \frac{k_1^2 k_3 A^2}{k_4^3}$

Brusselator IV - perturbations at the homogeneous fixed point

$$\dot{u} = \Delta u + 1 - (1+b)u + \lambda u^2 v$$
$$\dot{v} = d\Delta v \qquad +bu - \lambda u^2 v$$

fixed point

$$u^* = 1, \;\; v^* = b/\lambda$$

$$u - u^* = u_k e^{ikx}$$
$$v - v^* = v_k e^{ikx}$$

Linearization

$$\dot{u}_k = (b - 1 - k^2)u_k + \lambda v_k$$
$$\dot{v}_k = -bu_k + (-\lambda - dk^2)v_k$$

$$u_k = u_k^{(0)} e^{\omega_k t}$$
$$v_k = v_k^{(0)} e^{\omega_k t}$$

$$\left\| \begin{matrix} \omega_k + 1 - b + k^2, & -\lambda \\ b & , \omega_k + \lambda + dk^2 \end{matrix} \right\| = 0$$

$$\omega_k = f(k, \lambda, b, d)$$

Brusselator V- eigenvalue analysis

$$\omega_k^2 + [1 + \lambda - b + (d+1)k^2]\omega_k + b\lambda + (1 - b + k^2)(\lambda + dk^2) = 0$$

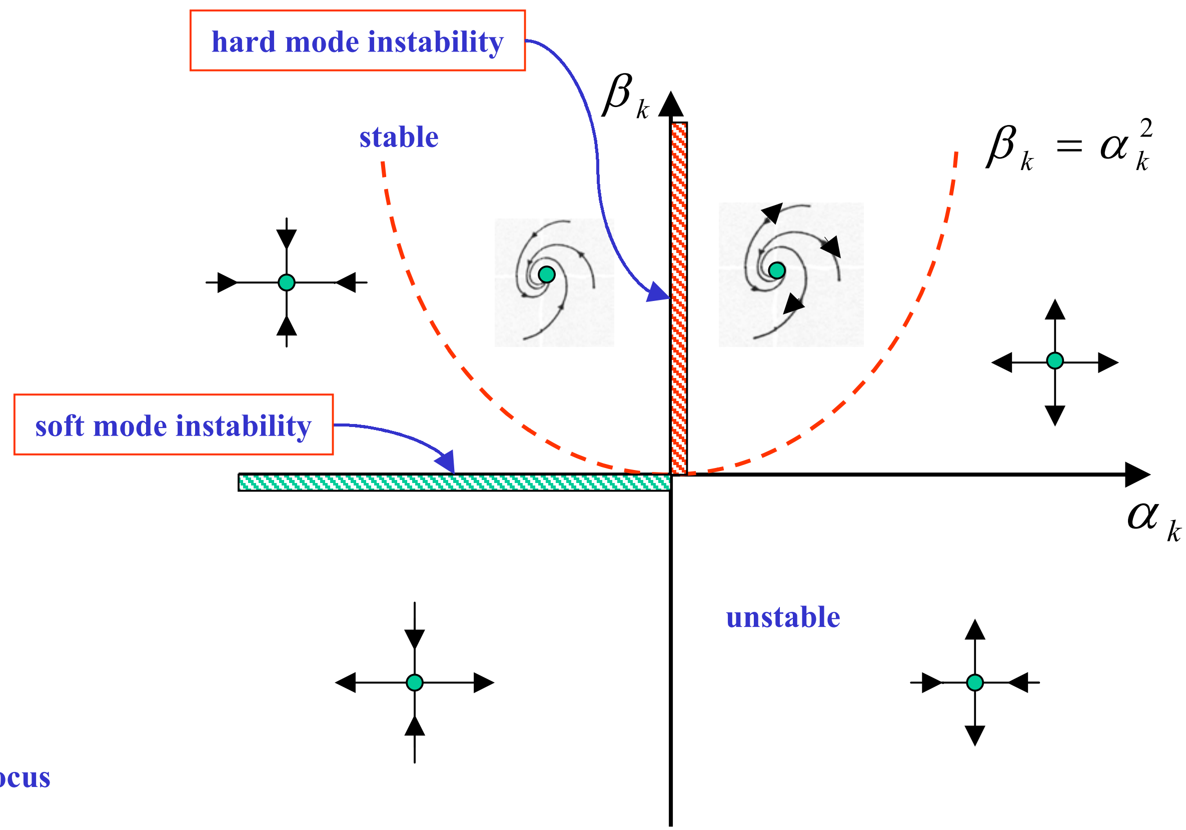
$$[1 + \lambda - b + (d+1)k^2] = -2\alpha_k \qquad b\lambda + (1 - b + k^2)(\lambda + dk^2) = \beta_k$$

$$\omega_k = \alpha_k \pm \sqrt{\alpha_k^2 - \beta_k}$$

$\beta_k < 0$ ← unstable

$\beta_k > 0, \ \alpha_k > 0$ ← unstable

$\beta_k > 0, \ \alpha_k < 0$ ← stable



$\beta_k > \alpha_k^2$ stable or unstable focus

$\alpha_k < 0$ $\alpha_k > 0$

Brusselator VI- hard mode instability

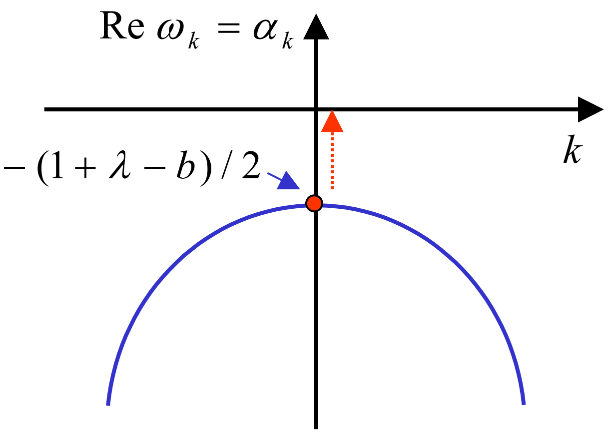
$$\omega_k^2 + [1 + \lambda - b + (d+1)k^2]\omega_k + b\lambda + (1 - b + k^2)(\lambda + dk^2) = 0$$

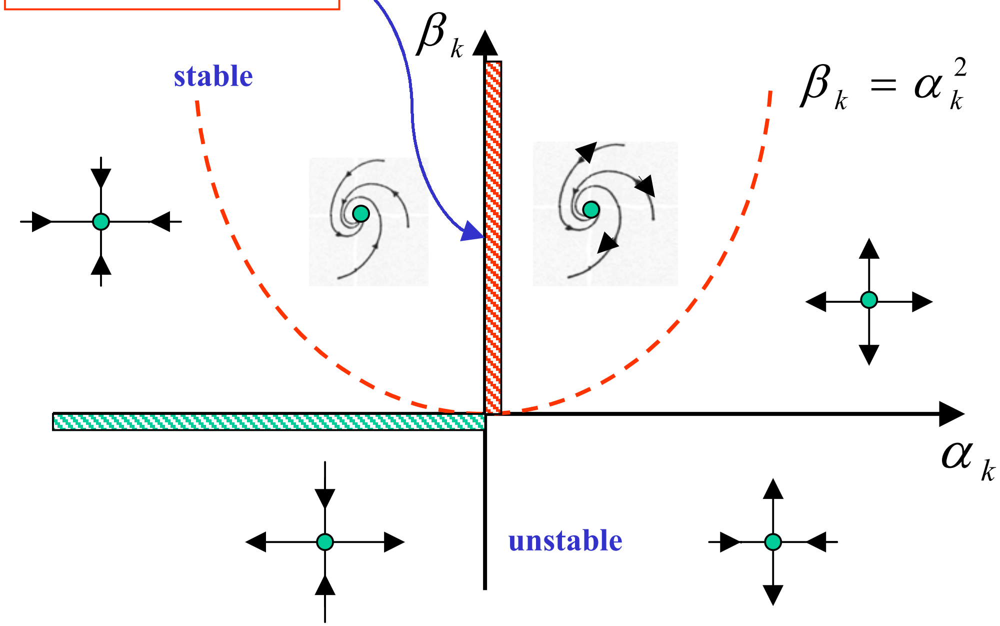
$$-2\alpha_k \qquad\qquad \beta_k$$

$$\omega_k = \alpha_k \pm \sqrt{\alpha_k^2 - \beta_k}$$

$$\beta_k > 0, \ \alpha_k \to 0^-$$

hard mode instability





(1) k=0 instability

(2) instability point $\lambda_c = b - 1$

(3) exists only for $b > 1$

$$\beta_{k=0} = \lambda > 0$$

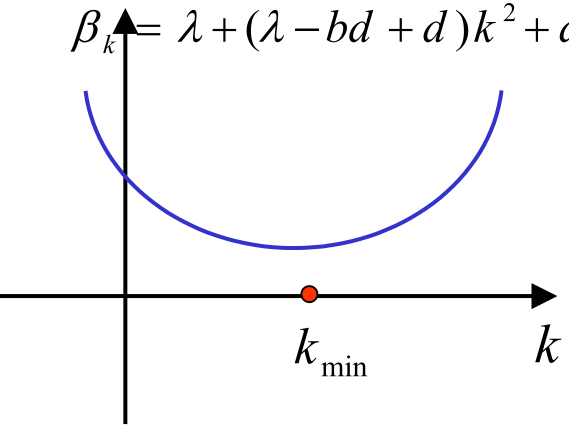
Brusselator VII- soft mode instability

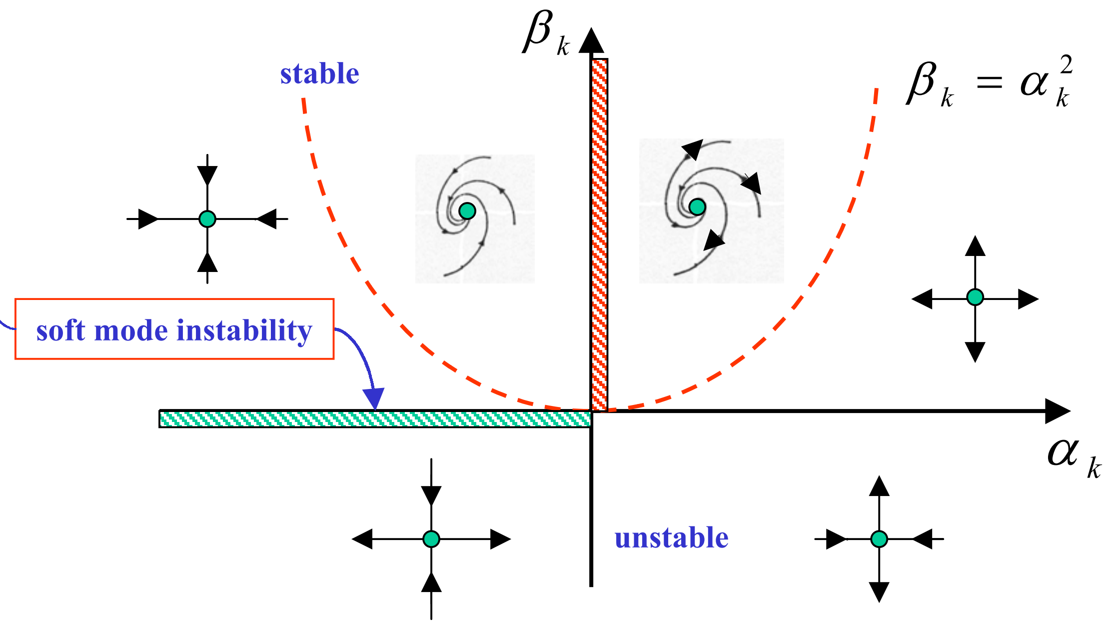
$$\omega_k^2 + \underbrace{[1+\lambda-b+(d+1)k^2]}_{-2\alpha_k}\omega_k + \underbrace{b\lambda+(1-b+k^2)(\lambda+dk^2)}_{\beta_k} = 0$$

$$\omega_k = \alpha_k \pm \sqrt{\alpha_k^2 - \beta_k}$$

$$\alpha_k < 0, \quad \beta_k \to 0^+$$

$$\beta_k = \lambda + (\lambda - bd + d)k^2 + dk^4$$





instability point

$$\beta_k = 0 \qquad \frac{\partial \beta_k}{\partial k^2} = 0$$

$$\lambda_c = d(\sqrt{b} - 1)^2$$

Soft mode instabtility first if

$$\lambda_c^{soft} > \lambda_c^{hard} \rightarrow d > \frac{\sqrt{b}+1}{\sqrt{b}-1} > 1 \rightarrow D_v > D_u$$

Emergence of spatial structures: Stability analysis

Linearization

$$\dot{n}_1 = D_1 \Delta n_1 + f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = D_2 \Delta n_2 + f_2(n_1, n_2, \lambda)$$

$$n_1^*, n_2^*$$

$$n_1 = n_1^* + \delta n_{1k} e^{ikx}$$
$$n_2 = n_2^* + \delta n_{2k} e^{ikx}$$

$$\begin{pmatrix} \delta \dot{n}_{1k} \\ \delta \dot{n}_{2k} \end{pmatrix} = \mathbf{A}_{\lambda}(k) \begin{pmatrix} \delta n_{1k} \\ \delta n_{2k} \end{pmatrix}$$

Stability matrix

Diagonalization

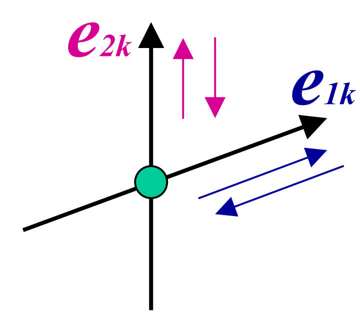
$$\tilde{\mathbf{A}}_{\lambda} \Rightarrow \begin{pmatrix} \omega_{1\lambda}(k) & 0 \\ 0 & \omega_{2\lambda}(k) \end{pmatrix}$$

Eigenvalues

Solution

$$\begin{pmatrix} \delta n_{1k} \\ \delta n_{2k} \end{pmatrix} = c_{1k} \mathbf{e}_{1k} e^{\omega_1(k)t} + c_{2k} \mathbf{e}_{2k} e^{\omega_2(k)t}$$

Eigenvectors



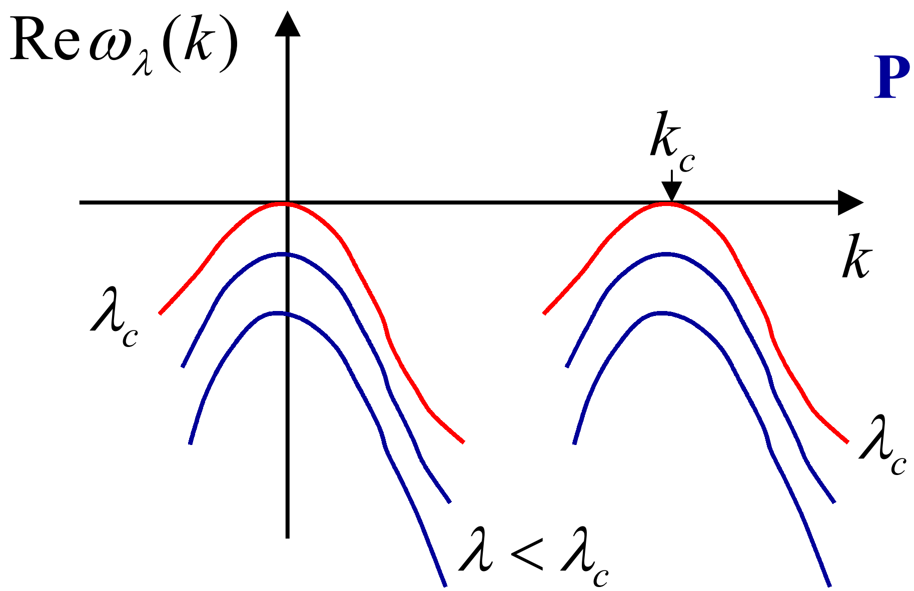
Critical slowing down and classification of instabilities

$$\omega_{\lambda 1,2}(k) \Rightarrow \omega_\lambda(k)$$

- with the largest real part

Instability: $\operatorname{Re}\omega_\lambda(k) \to 0^-$

$$\lambda \to \lambda_c$$



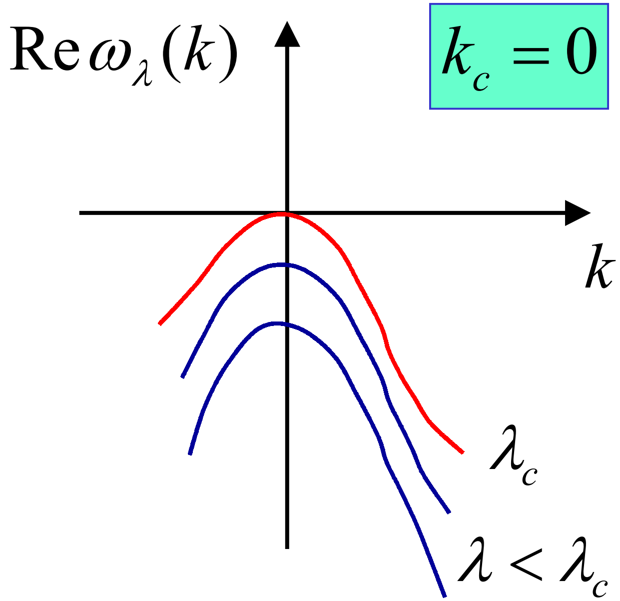
Possibilities:

$$k_c \begin{cases} = 0 \\ \neq 0 \end{cases}$$

$$\operatorname{Im}\omega_{\lambda_c}(k_c) \begin{cases} = 0 & \text{soft} \\ \neq 0 & \text{hard} \end{cases}$$

Classification of instabilities - emerging structures

$k_c = 0$

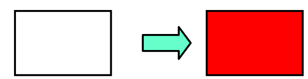


$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$

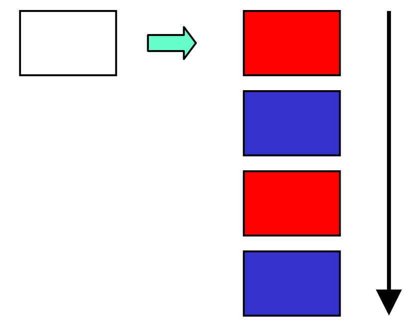
$\mathrm{Im}\,\omega_{\lambda_c}(k_c) \neq 0$

spatially homogeneous

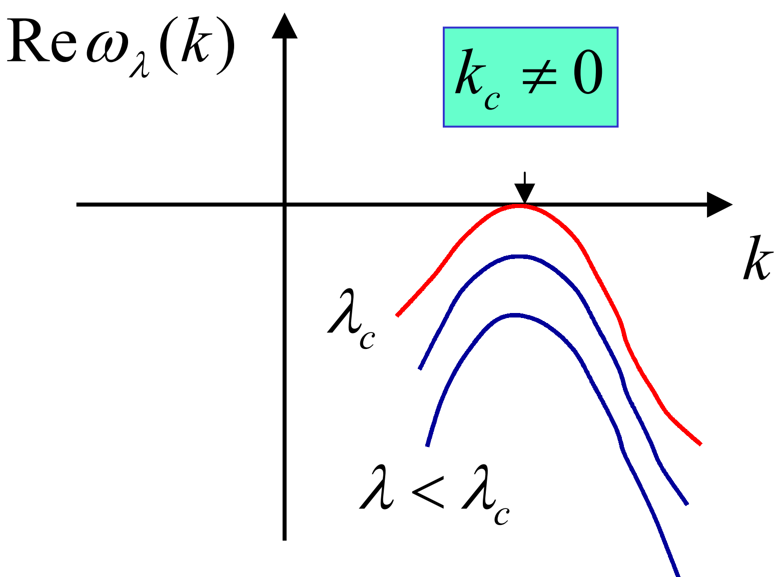
stationary



limit cycle

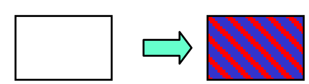


$k_c \neq 0$

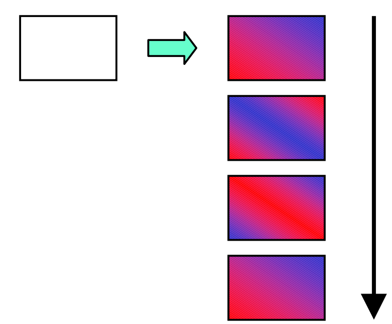


spatially structured

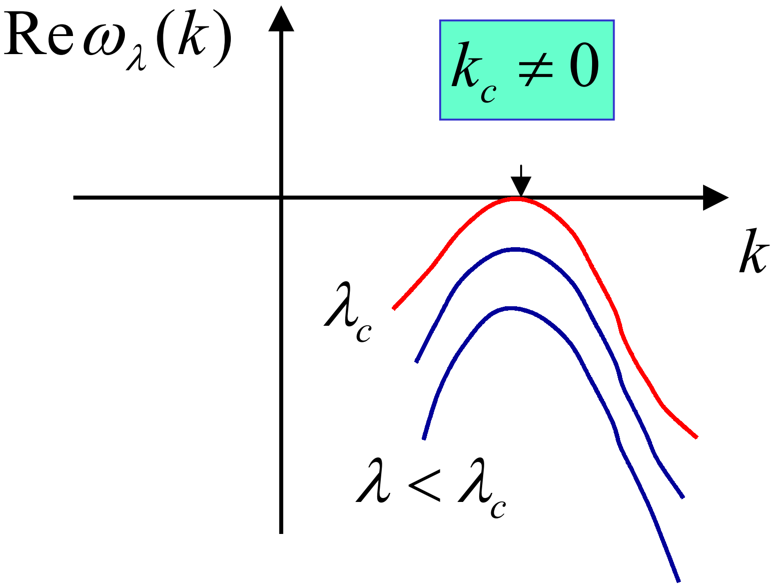
stationary



time- and space-dependent

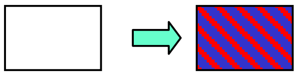


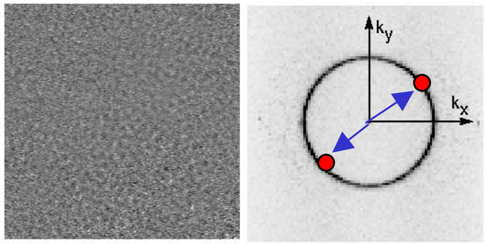
Stationary structures emerging in d=2 homogeneous systems



$k_c \neq 0$

$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$

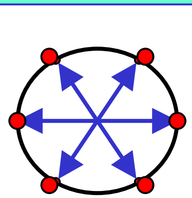


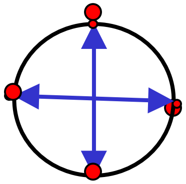


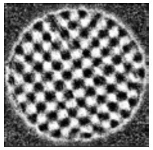
d=2
isotropy

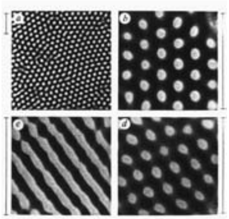
$$\delta n_{\vec{k}} \sim a_{\vec{k}} e^{i\vec{k}\vec{x}+\omega_\lambda(k)t}$$

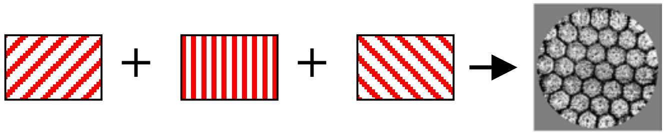
$$\omega_\lambda(\vec{k}) \Rightarrow \omega_\lambda(k)$$

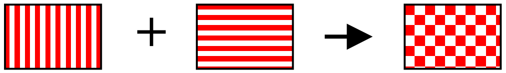












Swinney et al.
Turing patterns