Pattern Formation: Appendix II

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Amplitude equations

(1) Critical slowing down and amplitude equations for the slow modes.

Landau-Ginzburg equation with real coefficients.
Symmetry considerations and linear combination of slow modes.
Boundary conditions - pattern selection by ramp.

(2) Weakly nonlinear analysis of the dynamics of patterns.

Secondary instabilities of spatial structures.
- Eckhaus instability.
- Zig-zag instability.
- Time dependent structures.

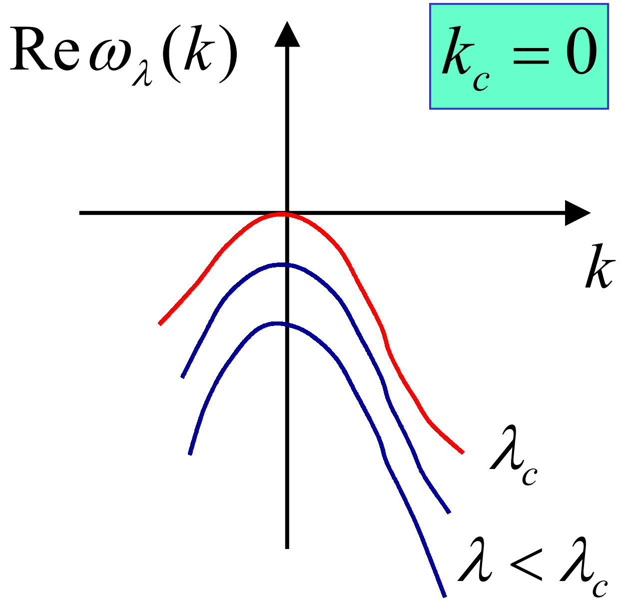
(3) Complex Landau-Ginzburg equation

Convective and absolute instabilities of patterns.
Benjamin-Feir instability - spatio-temporal chaos.
One-dimensional coherent structures. Noise sustained structures.

Literature

M. C. Cross and P. C. Hohenberg, **Pattern Formation Outside of Equilibrium**, Rev. Mod. Phys. 65, 851 (1993).

Classification of instabilities - emerging structures



$k_c = 0$

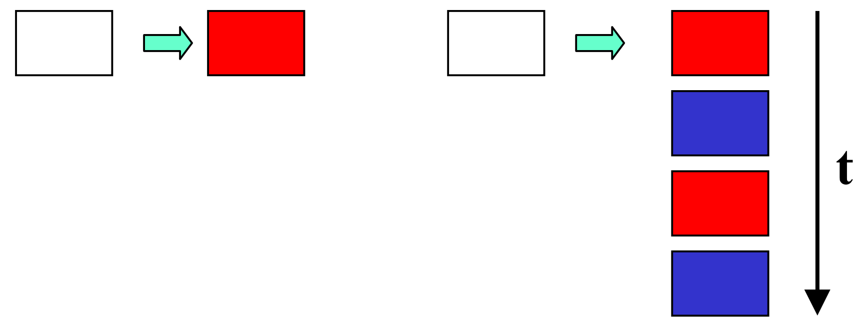
$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$

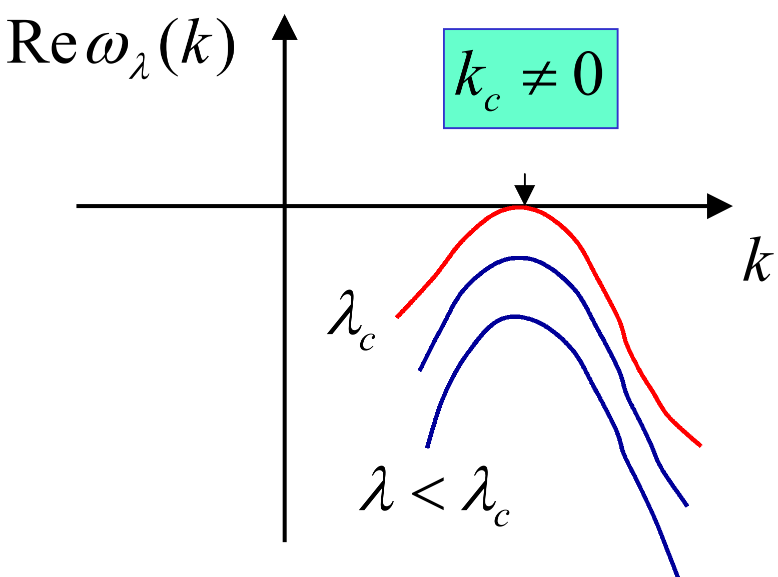
$\mathrm{Im}\,\omega_{\lambda_c}(k_c) \neq 0$

spatially homogeneous

stationary

limit cycle



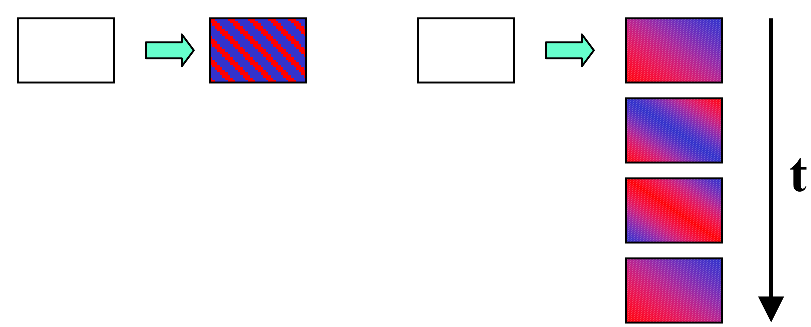


$k_c \neq 0$

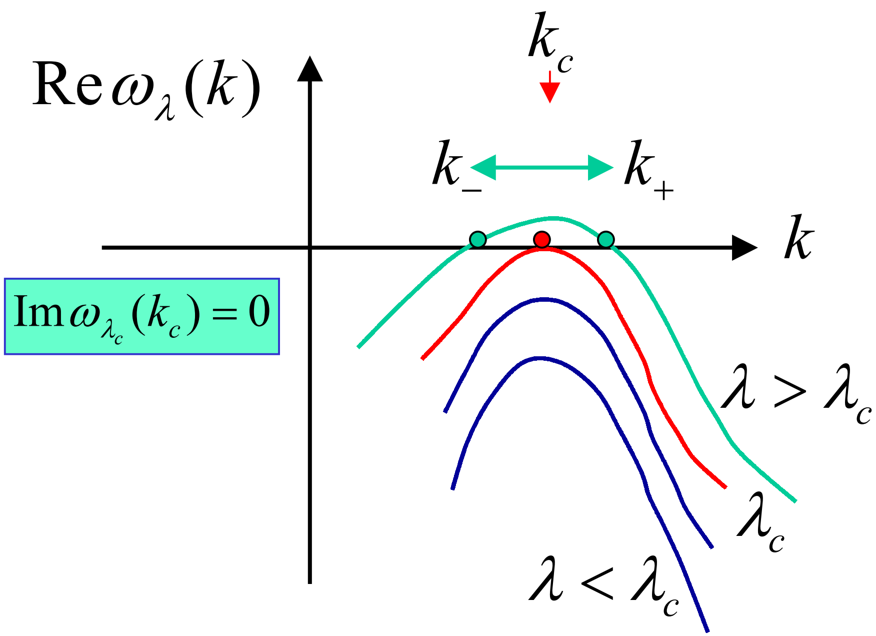
spatially structured

stationary

time- and space-dependent

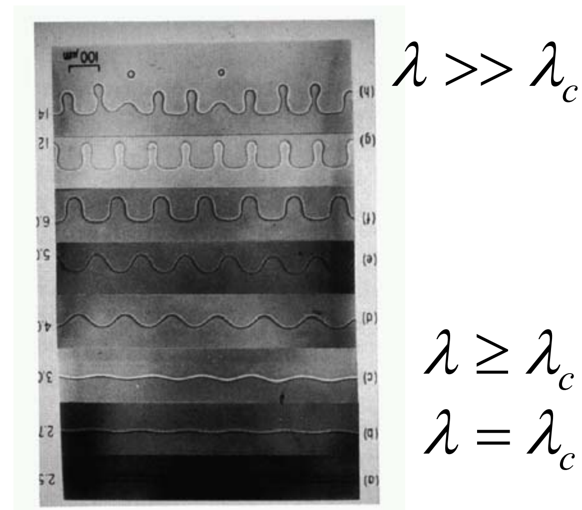


Beyond the instability: Amplitude equation for slow modes



$\lambda > \lambda_c$ Band of unstable modes

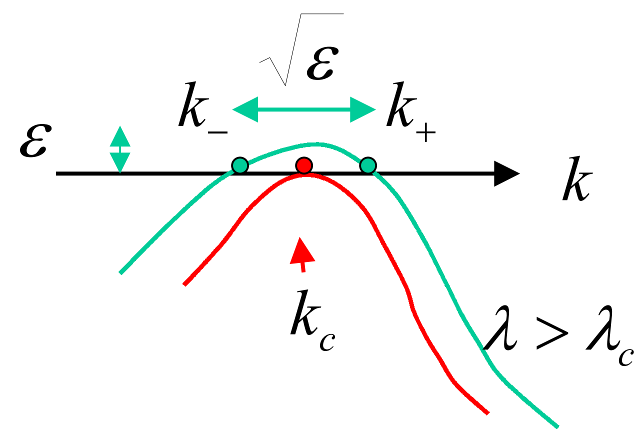
What is the steady state?



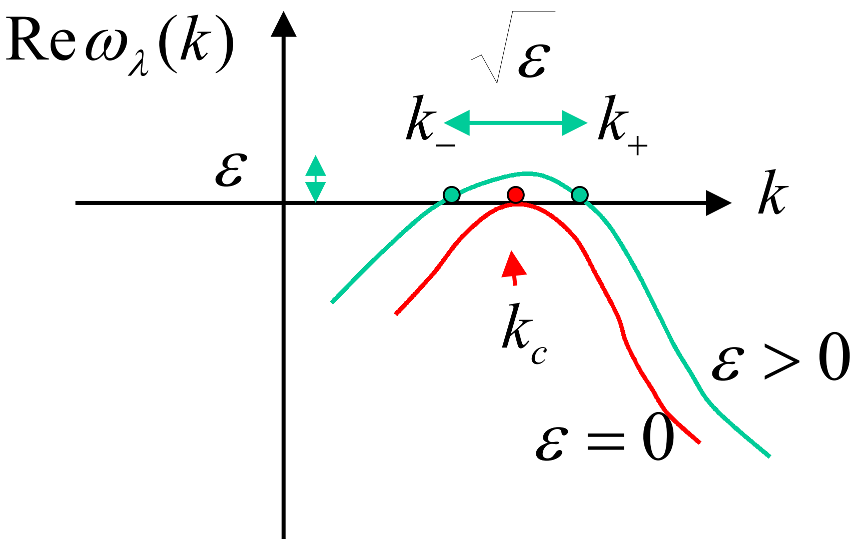
$\mathrm{Re}\,\omega_\lambda(k)$ smooth function of

$$\omega_\lambda(k) \approx \underbrace{\lambda - \lambda_c}_{\varepsilon} - a(k - k_c)^2$$

ε **control parameter from now on**



Amplitude equation: Characteristic lengths and times



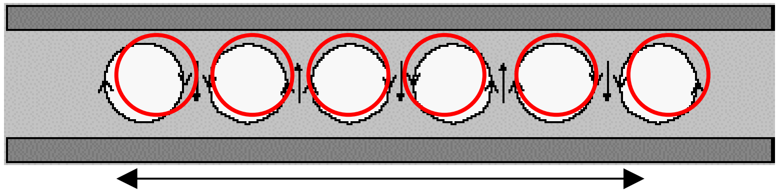
$\varepsilon > 0$ Band of unstable modes

$$\omega_\lambda(k) \approx \varepsilon - a(k - k_c)^2$$

$$n(x,t) - n^* = \int_{-\infty}^{\infty} dk\, \tilde{n}_k\, e^{ikx + \omega_\lambda(k)t}$$

$$\approx e^{ik_c x} \underbrace{\int_{k_-}^{k_+}}_{\sim\sqrt{\varepsilon}} dk\, \tilde{n}_k\, e^{i\overbrace{(k-k_c)}^{\sim\sqrt{\varepsilon}}x + \overbrace{\omega_\lambda(k)}^{\sim\varepsilon}t}$$

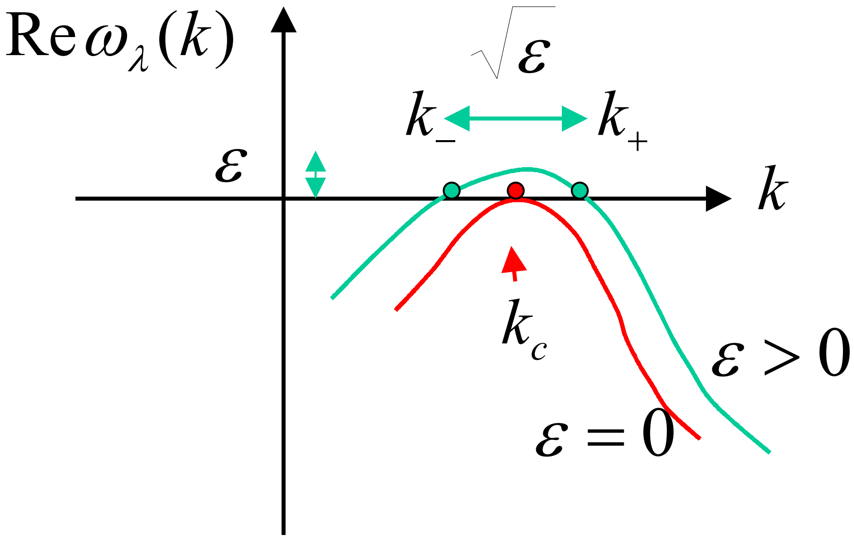
$$\approx e^{ik_c x}\sqrt{\varepsilon}\, A_0(\sqrt{\varepsilon}x, \varepsilon t)$$



$\xi \sim 1/\sqrt{\varepsilon}$

Variation of the amplitude of the periodic structure on lenghtscale $\xi \sim 1/\sqrt{\varepsilon}$ **and on timescale** $\tau \sim 1/\varepsilon$.

Amplitude equation



$\varepsilon > 0$ Band of unstable modes

$$\omega_\lambda(k) \approx \varepsilon - a(k - k_c)^2$$

$$\xi \sim 1/\sqrt{\varepsilon} \qquad \tau \sim 1/\varepsilon$$

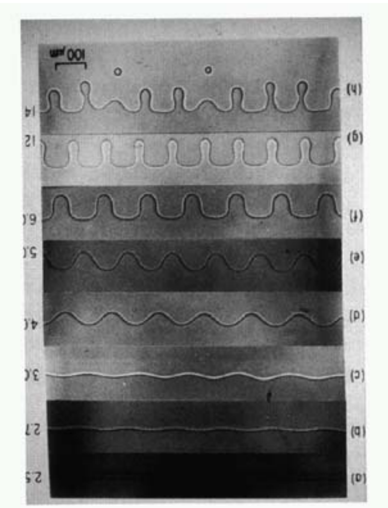
$$n(x,t) - n^* \approx e^{ik_c x}\sqrt{\varepsilon}\,A_0(\sqrt{\varepsilon}x, \varepsilon t) \equiv e^{ik_c x} A(x,t)$$

Plug it in the original equation and expand.

Amplitude equation:

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

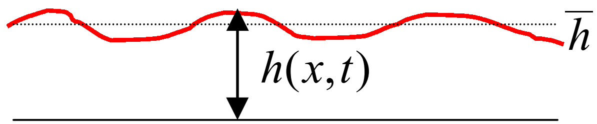
Amplitude eq.: Derivation from the Swift-Hohenberg equation



$\varepsilon >> 0$

$\varepsilon \geq 0$

$\varepsilon = 0$



$u(x,t) = h(x,t) - \bar{h}$

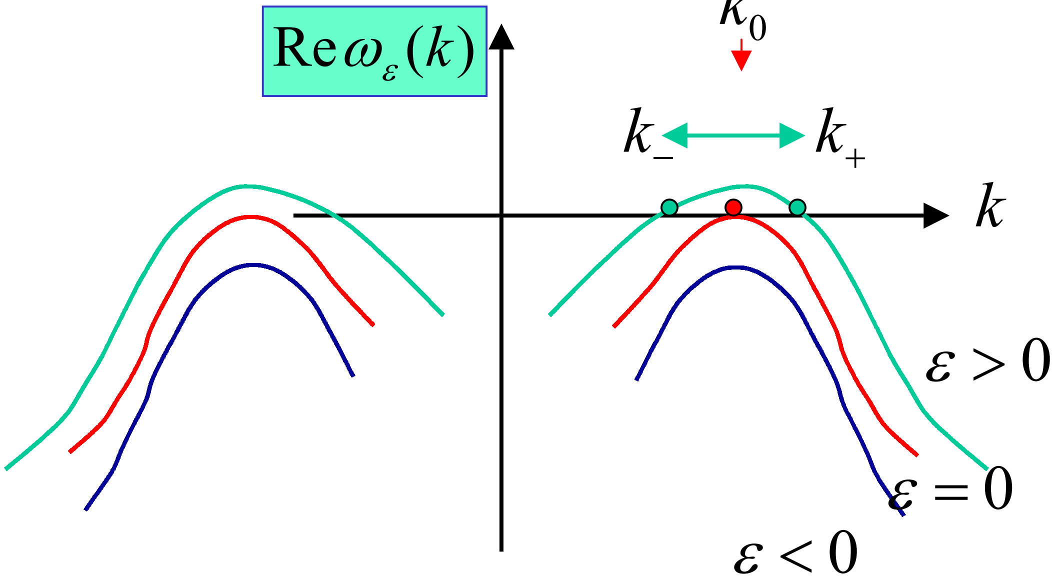
near instability:

$$\partial_t u = \varepsilon u - (\Delta + k_0^2)^2 u - u^3$$

linearization:

$$\partial_t u_k = [\varepsilon - (k^2 - k_0^2)^2] u_k$$

$\omega_\varepsilon(k)$



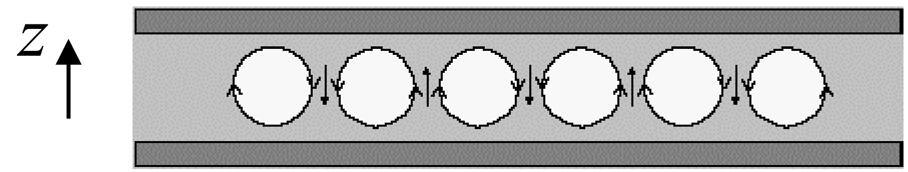
$\mathrm{Im}\,\omega_0(k_0) = 0$

Amplitude equation: Simple solutions

small $\varepsilon > 0$

$$n(x,t) - n^* \approx e^{ik_c x}\sqrt{\varepsilon}A_0(\sqrt{\varepsilon}x, \varepsilon t) \equiv e^{ik_c x}A(x,t)$$

z-component of the velocity v_z



$$v_z = A\cos(k_c x)$$

$A =$ const.

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A \longrightarrow A = \pm\sqrt{\varepsilon}$$

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A \longrightarrow A = \sqrt{\varepsilon}A_0(\sqrt{\varepsilon}x, \varepsilon t)$$

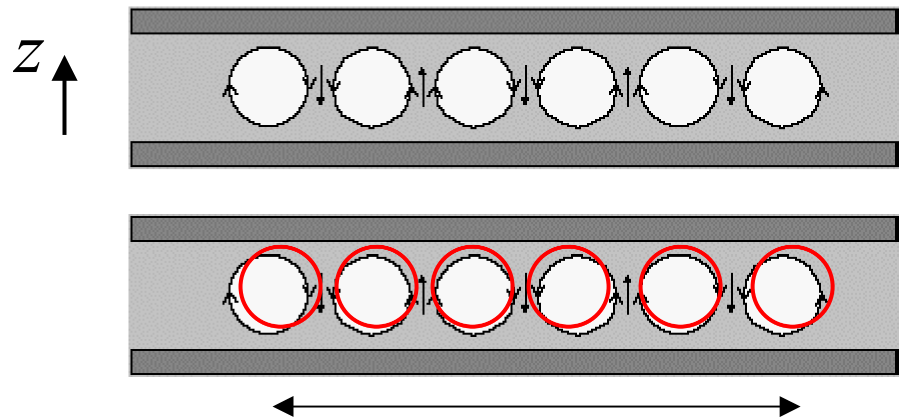
general solution

Amplitude equation: Why is it so general?

small $\varepsilon > 0$

$$n(x,t) - n^* \approx e^{ik_c x}\sqrt{\varepsilon}\, A_0(\sqrt{\varepsilon}\, x, \varepsilon t) \equiv e^{ik_c x} A(x,t)$$

z-component of the velocity v_z



$$\xi \sim 1/\sqrt{\varepsilon}$$

Linear stability changes with ε changing sign

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

lowest order in spatial derivatives preserving ↔ symmetry

Slow, large-scale motion around the stationary structure should not depend on the position of the underlying structure

$$e^{ik_c x} A(x,t) \to e^{ik_c (x+l)} A(x,t)$$

$$= e^{ik_c x} e^{ik_c l} A(x,t) \to e^{ik_c x} B(x,t)$$

Amplitude equation: What can we get out of it?

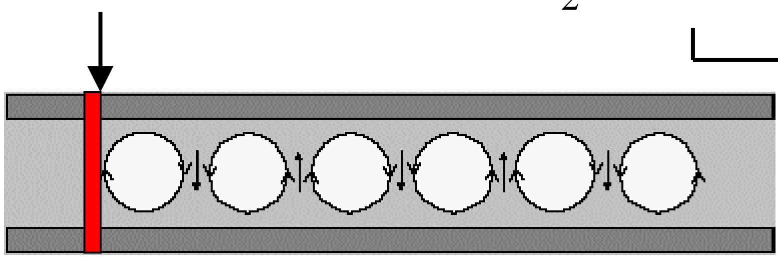
$$v_z = e^{ik_c x} A(x,t)$$

$$\varepsilon > 0$$

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

$$v_z = 0$$

$$v_z = \sqrt{\varepsilon} e^{ik_c x}$$



stationary state

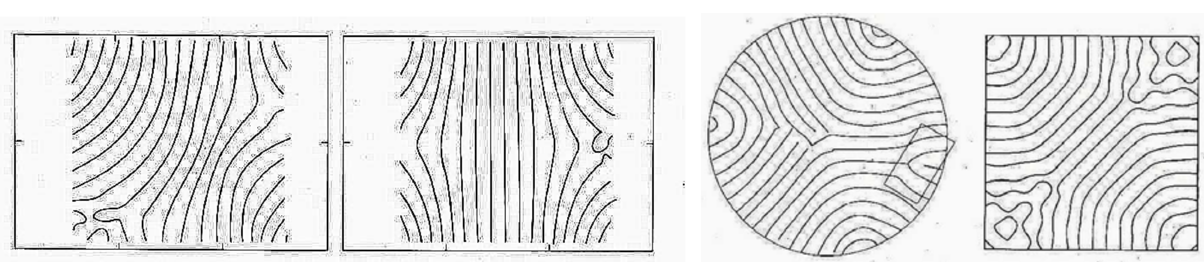
$$A = 0 \longleftrightarrow A(x \to \infty) = \sqrt{\varepsilon}$$

$$\xi \sim 1/\sqrt{\varepsilon}$$

boundary conditions

$$0 = \varepsilon A + \frac{d^2 A}{d x^2} - |A|^2 A$$

$$A = \sqrt{\varepsilon}\, \mathrm{th}\left(\sqrt{\frac{\varepsilon}{2}}\, x\right)$$

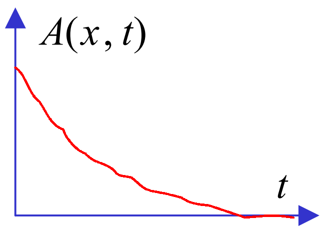


(a) (b) (c) (d)

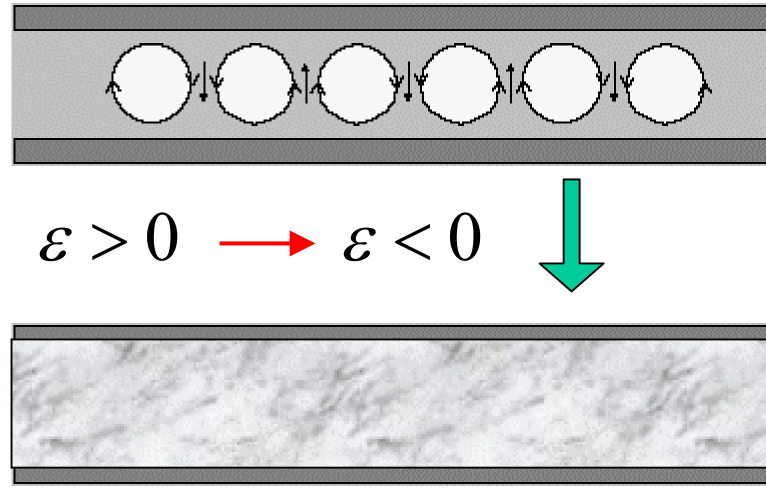
Scale of A and x are determined

Amplitude equation: Fixing the time-scale

Quenching from ordered into disordered state



$$v_z = e^{ik_c x} A(x,t) \rightarrow 0$$



$$\varepsilon > 0 \rightarrow \varepsilon < 0$$

Amplitude equation should be still good

$$\frac{\partial A}{\partial t} = -|\varepsilon| A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

$(A \rightarrow 0) \qquad \approx 0$

$$A(x,t) = a_k(t) e^{ikx}$$

$$\dot{a}_k = -(\varepsilon + k^2) a_k$$

Relaxation time

$$\tau_k = \frac{1}{\varepsilon + k^2}$$

Amplitude equation: Secondary instabilities I

$\varepsilon > 0$

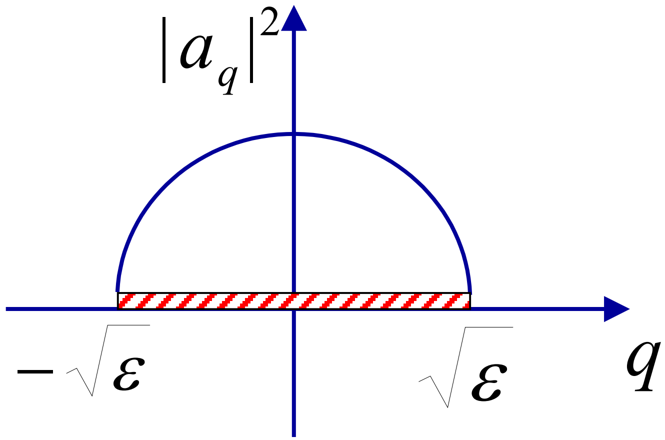
Large number of possible stationary states

$$A = a_q e^{iqx}$$

$$0 = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

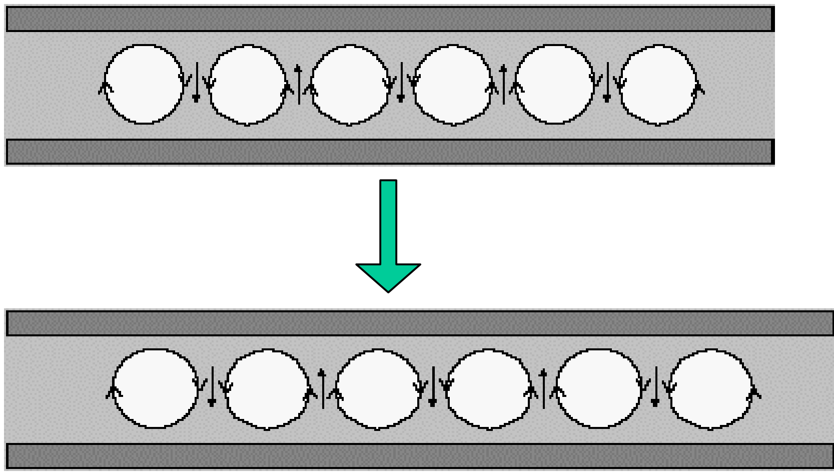
$$\left(\varepsilon - q^2 - |a_q|^2\right) a_q = 0$$

$$|a_q|^2 = \varepsilon - q^2$$



Meaning: Shift in the wavelength of the pattern

$$v_z = e^{ik_c x} a_0$$



$$v_z = e^{ik_c x} A = e^{i(k_c + q)x} a_q$$

Amplitude equation: Secondary instabilities II

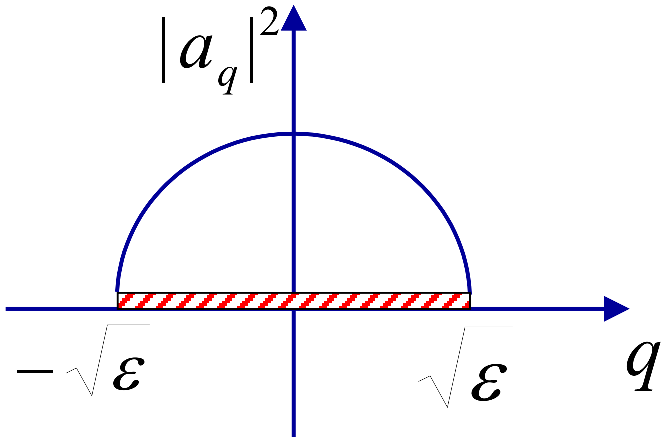
$\varepsilon > 0$

Large number of possible stationary states

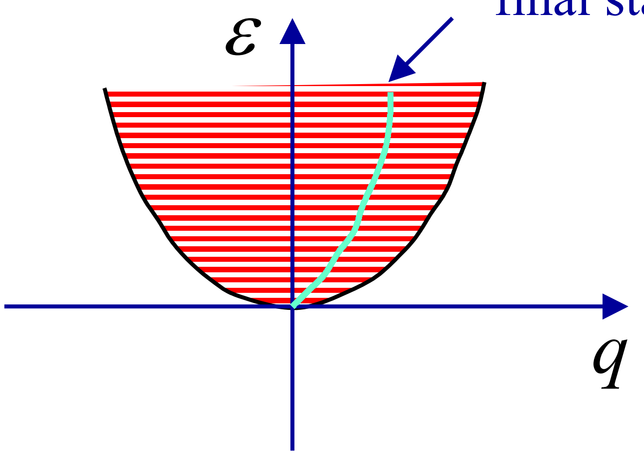
$$0 = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

$$A = a_q e^{iqx}$$

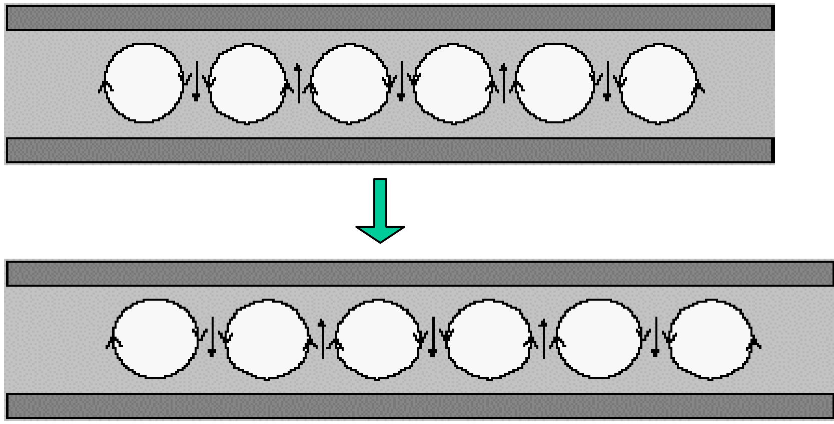
$$v_z = e^{ik_c x} A = e^{i(k_c + q)x} a_q$$



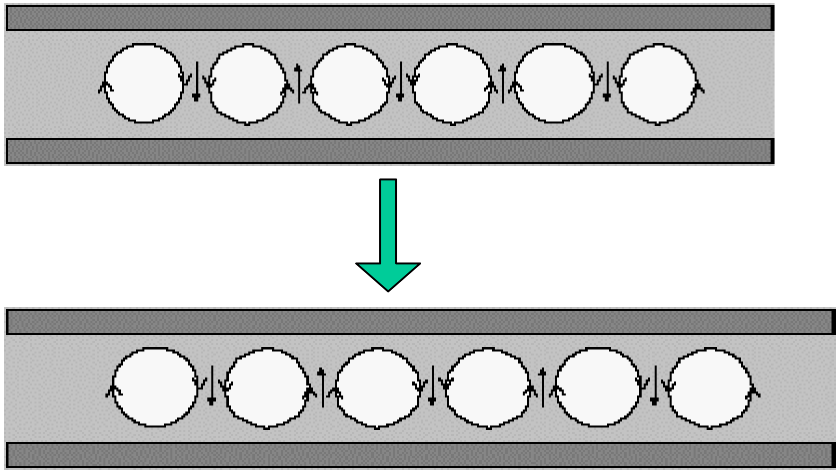
final state



Phase winding solutions



Amplitude equation: Secondary instabilities III



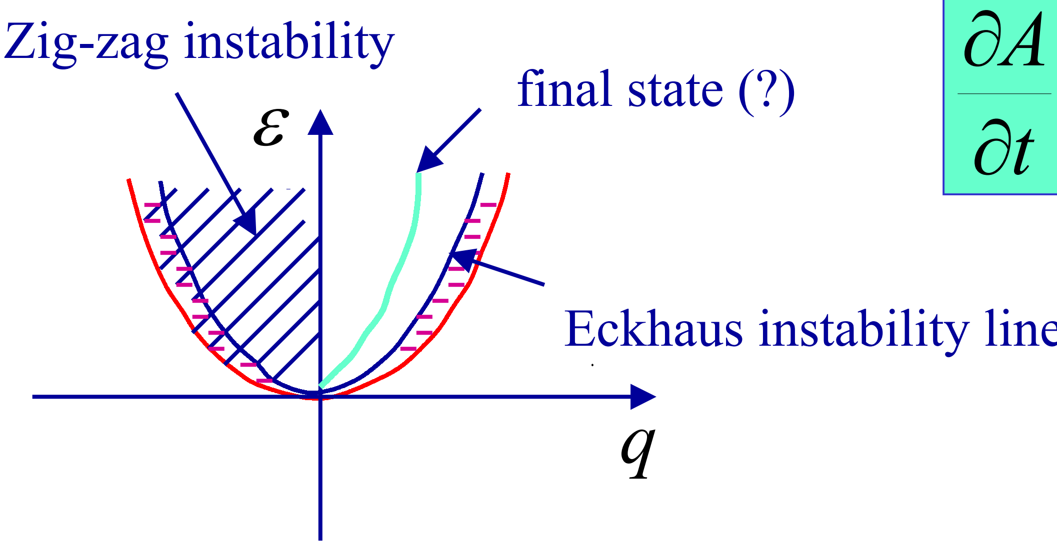
$$v_z = e^{ik_c x} A = e^{i(k_c+q)x} a_q$$

Stability analysis:

$$A = (a_q + \delta a) e^{iqx}$$

$$\delta a = \delta a(x, y, t)$$

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$



Amplitude eq.: Secondary instabilities: Phase diffusion

Y. Pomeau, P. Manneville

$$v_z = e^{ik_c x} A$$

$$A = (a_q + \delta a) e^{i(qx+\varphi)}$$

$\varphi = const$

does not decay

decays on timescale

$\tau_a \sim 1/\varepsilon$

$\varphi \sim b \sin Qx$

decays as

$\tau_Q \sim 1/Q^2$

$\tau_a << \tau_Q$

$$\delta a = f(\varphi, \partial_x \varphi)$$

Stability analysis:

$$\partial_t A = \varepsilon A + \partial_x^2 A - |A|^2 A$$

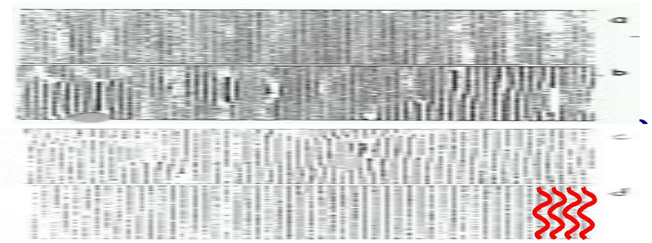
$$\partial_t \varphi = \frac{\varepsilon - 3q^2}{\varepsilon - q^2} \partial_x^2 \varphi$$

Eckhaus instability line: phase diffusion becomes unstable:

$$q_{\pm} = \sqrt{\varepsilon / 3}$$

Amplitude equation for A(x,y,t): Secondary instabilities

$$v_z = e^{ik_c x} A(x,y,t)$$



$$u = \varepsilon^{1/2} A_0(\varepsilon^{1/2} x, \varepsilon^{1/4} y, \varepsilon t)\Phi(x) + \ldots$$

$$\partial_t A = \varepsilon A + \left(\partial_x + \frac{i}{2k_c}\partial_y\right)^2 A - |A|^2 A$$

Stability analysis:

$$A = (a_q + \delta a) e^{i(qx+\varphi)}$$

$$\varphi = \varphi(x,y,t)$$

$$\delta a = f(\varphi, \partial_x \varphi)$$

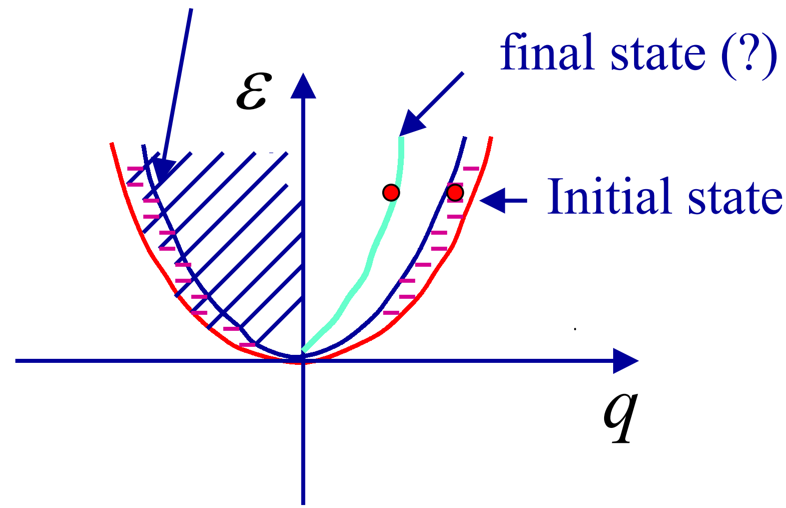
$$\partial_t \varphi = \frac{\varepsilon - 3q^2}{\varepsilon - q^2}\partial_x^2 \varphi + \frac{q}{2k_c}\partial_y^2 \varphi$$

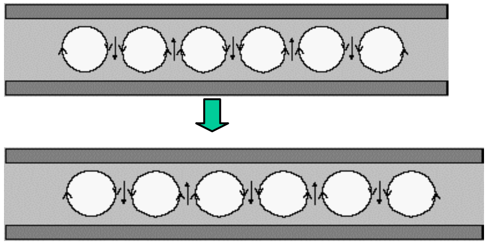
Zigzag instability: $q < 0$

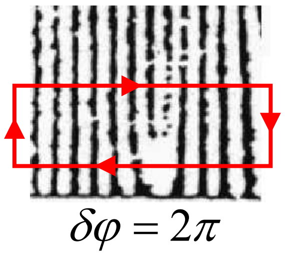
Dynamics of secondary instabilities: Topological defects

$$v_z = e^{ik_c x} A$$

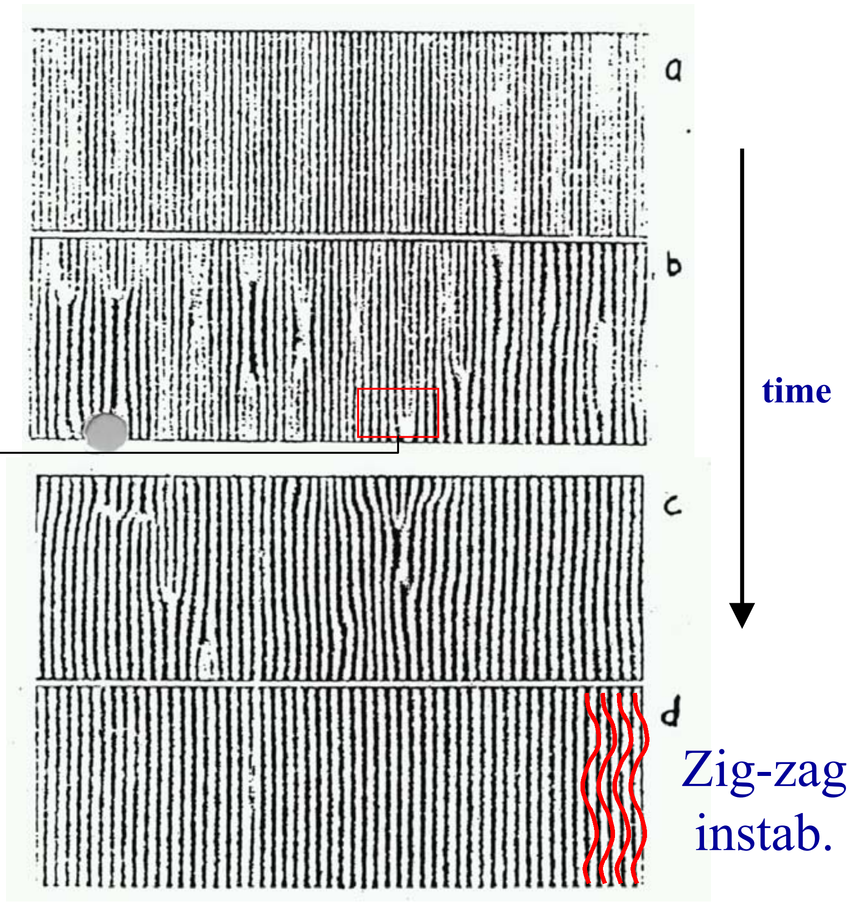
$$A = (a_q + \delta a) e^{i(qx+\varphi)}$$



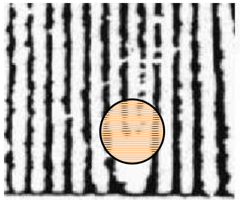




Not consistent with smooth change



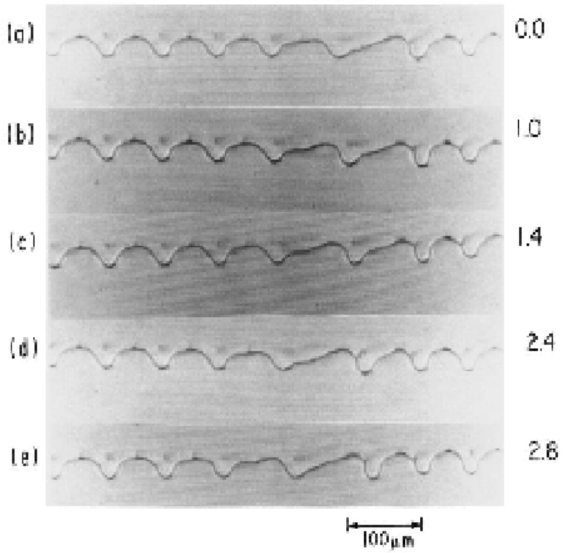
$$A \to 0$$



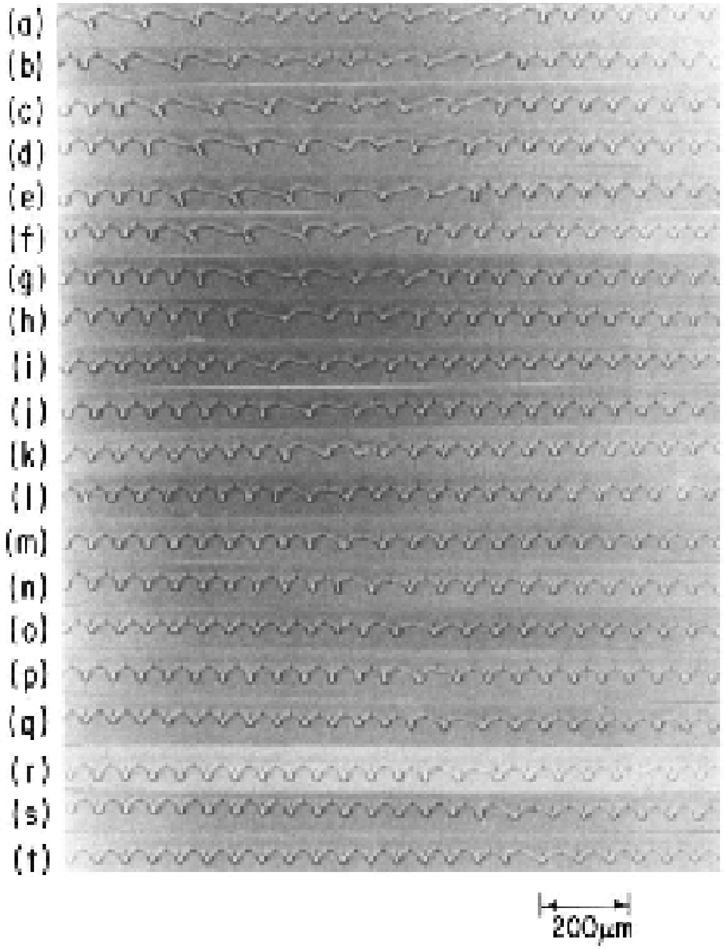
Translating structures

A.J. Simon, J. Bechofer, A. Libchaber

Isotropic-nematic transition



Solitary wave to moving to the left



Collision of two solitary waves

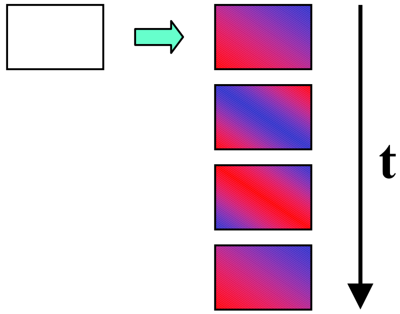
$$n(x,t) - n^* \approx e^{i(k_c x - \omega_c t)} \sqrt{\varepsilon} A_0(\sqrt{\varepsilon} x, \varepsilon t) \equiv A(x,t)\Phi(k_c x - \omega_c t)$$

Complex Landau-Ginzburg equation

$k_c \neq 0, \quad \mathrm{Im}\,\omega_{\lambda_c}(k_c) \neq 0$

$$n(x,t) - n^* \approx e^{i[k_c x - \mathrm{Im}\,\omega(k_c)t]} A(x,t)$$

time- and space-dependent

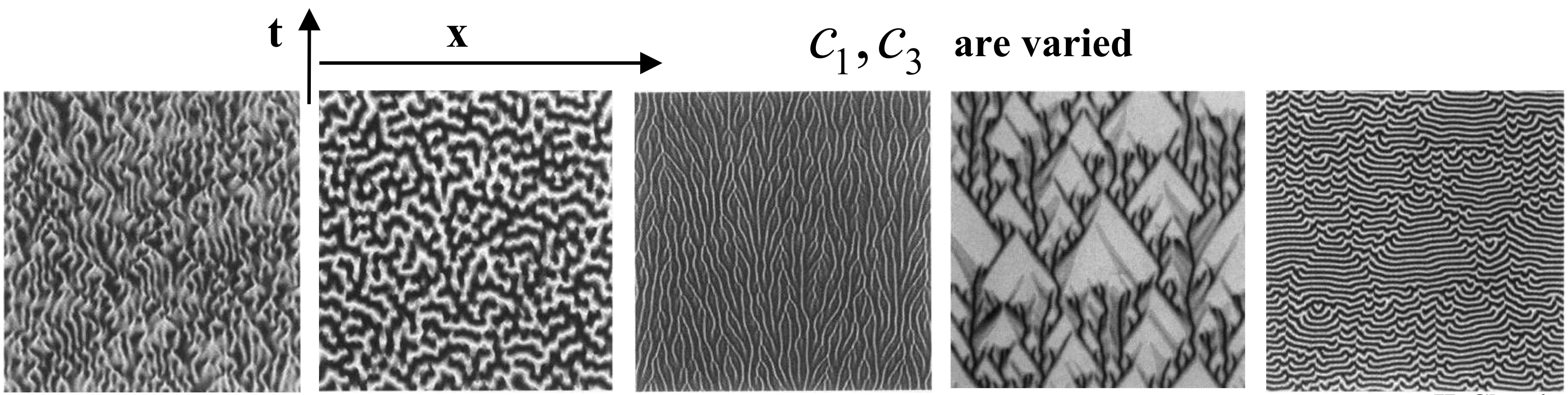


$$v = \mathrm{Im}\,\omega(k_c)/k_c$$

$$\frac{\partial A}{\partial t} + v\frac{\partial A}{\partial x} = \varepsilon A + (1+ic_1)\frac{\partial^2 A}{\partial x^2} - (1-ic_3)|A|^2 A$$

Velocity of the wave

c_1, c_3 **are varied**



H. Chate'

CLG equation: Secondary Instabilities

$$\partial_t A = \varepsilon A + (1+ic_1)\partial_x^2 A - (1-ic_3)|A|^2 A$$

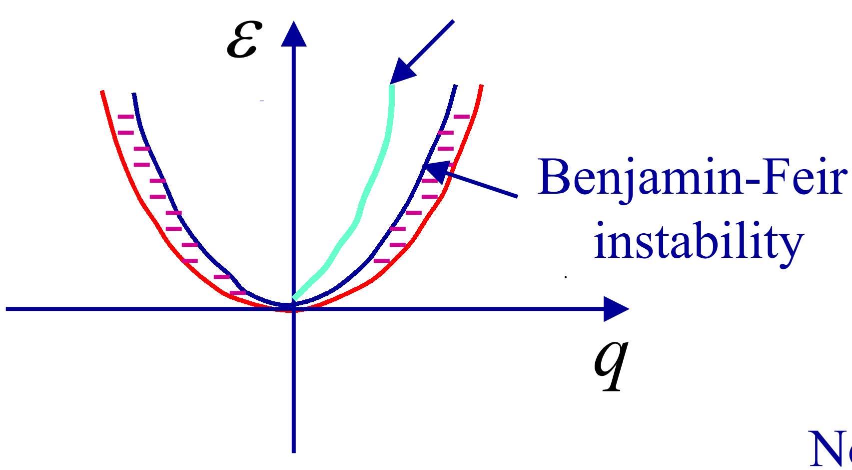
Phase winding solutions

$$A = a_{\omega,q} e^{i(qx-\omega t)}$$

$$\omega = c_1 q^2 - c_3 |a|^2$$

$$q^2 = \varepsilon - |a|^2$$

Linear stability



c_1, c_3 increased ⇨ linerly stable region decreases

$$c_1 c_3 > 1$$

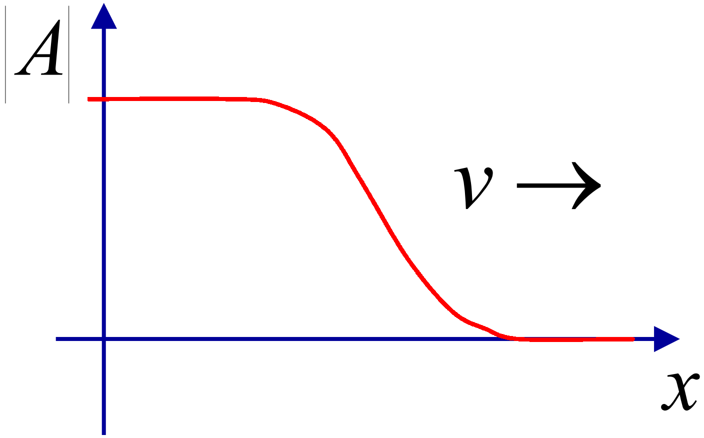
Newell criterion

No linerly stable region exists.

CLG equation: Coherent structures

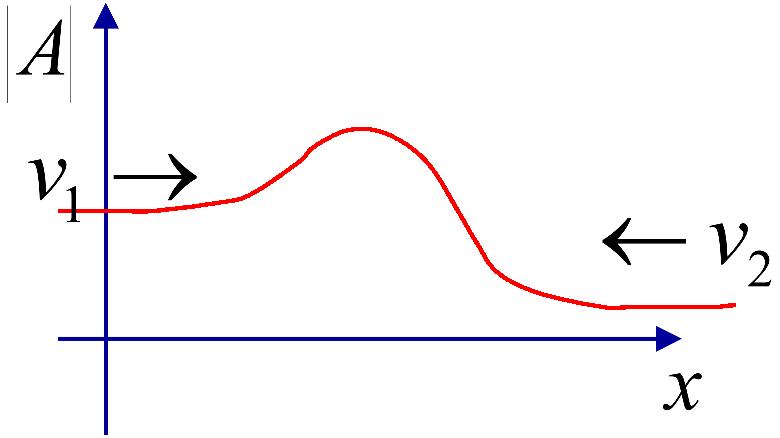
$$\partial_t A = \varepsilon A + (1+ic_1)\partial_x^2 A - (1-ic_3)|A|^2 A$$

Front



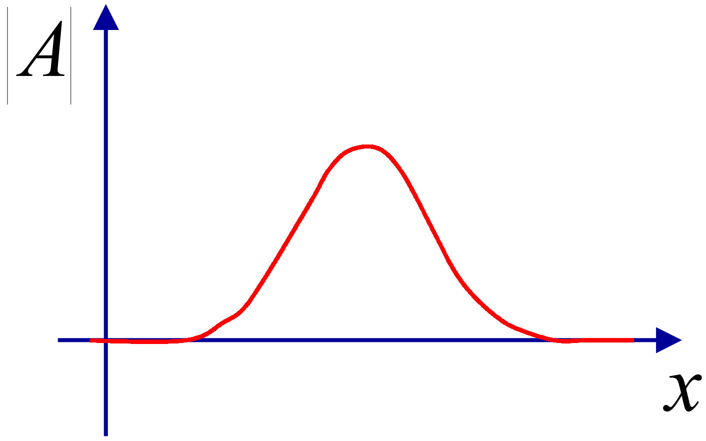
v = ? Pattern left behind?

Source



Two different phase-winding solutions

Pulse



Sink