Pattern Formation: Appendix III

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Patterns from moving fronts

(1) Importance of moving fronts: Patterns are manufactured in them.

Examples: Crystal growth, DLA, reaction fronts.
Dynamics of interfaces separating phases of different stability.
Classification of fronts: pushed and pulled.

(2) Invasion of an unstable state.

Velocity selection.
Example: Population dynamics.
Stationary point analysis of the Fisher-Kolmogorov equation.
Wavelength selection.
Example: Cahn-Hilliard equation and coarsening waves.

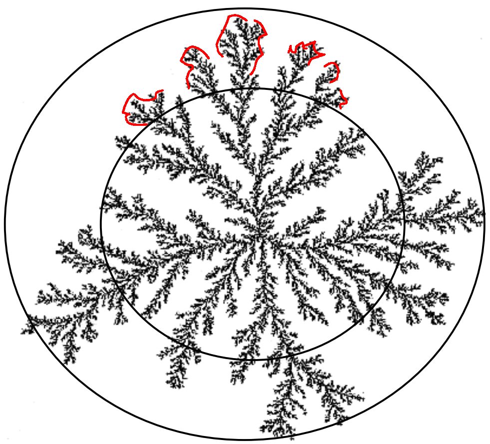
(3) Diffusive fronts.

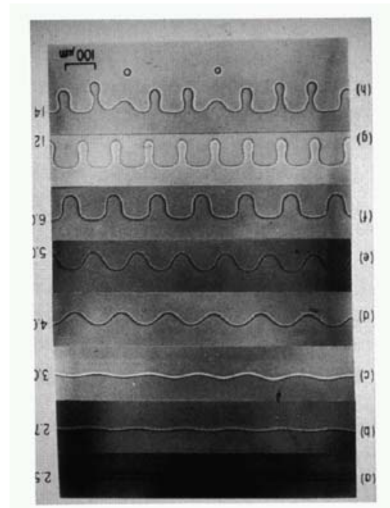
Liesegang phenomena (precipitation patterns in the wake of diffusive reaction fronts - a problem of distinguishing the general and particular).

Literature

M. C. Cross and P. C. Hohenberg, **Pattern Formation Outside of Equilibrium**, Rev. Mod. Phys. 65, 851 (1993).

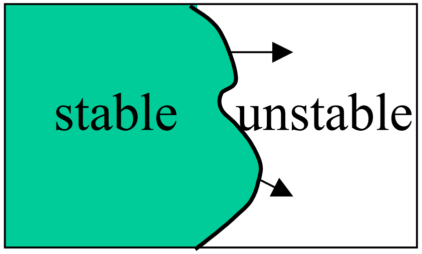
Fronts separating stable and unstable phases





crystallization fronts
chemical reaction fronts

The problems:

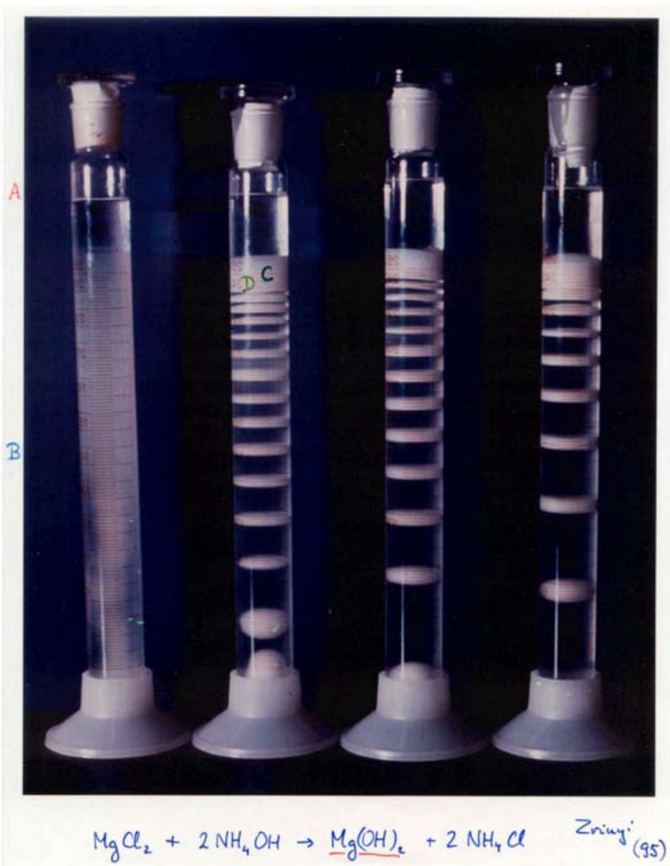


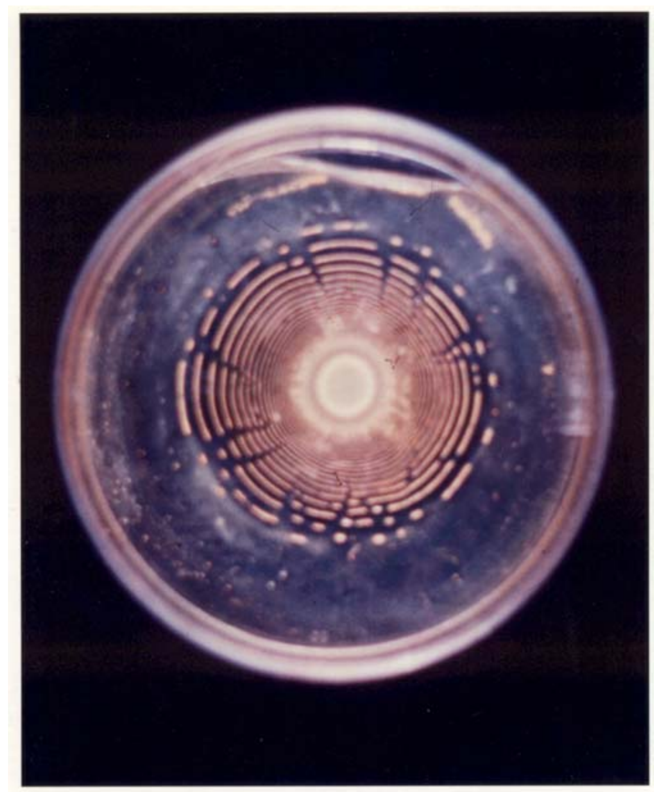
(1) What is the speed of the front?

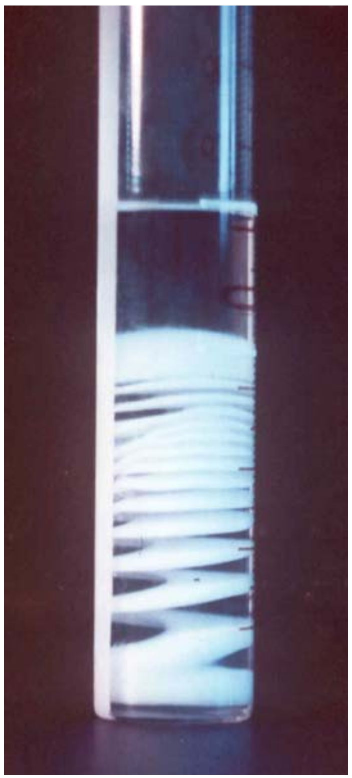
(2) Is there any nontrivial structure
in the wake of the front?

Liesegang patterns

An example of pattern formation in the wake of a moving front

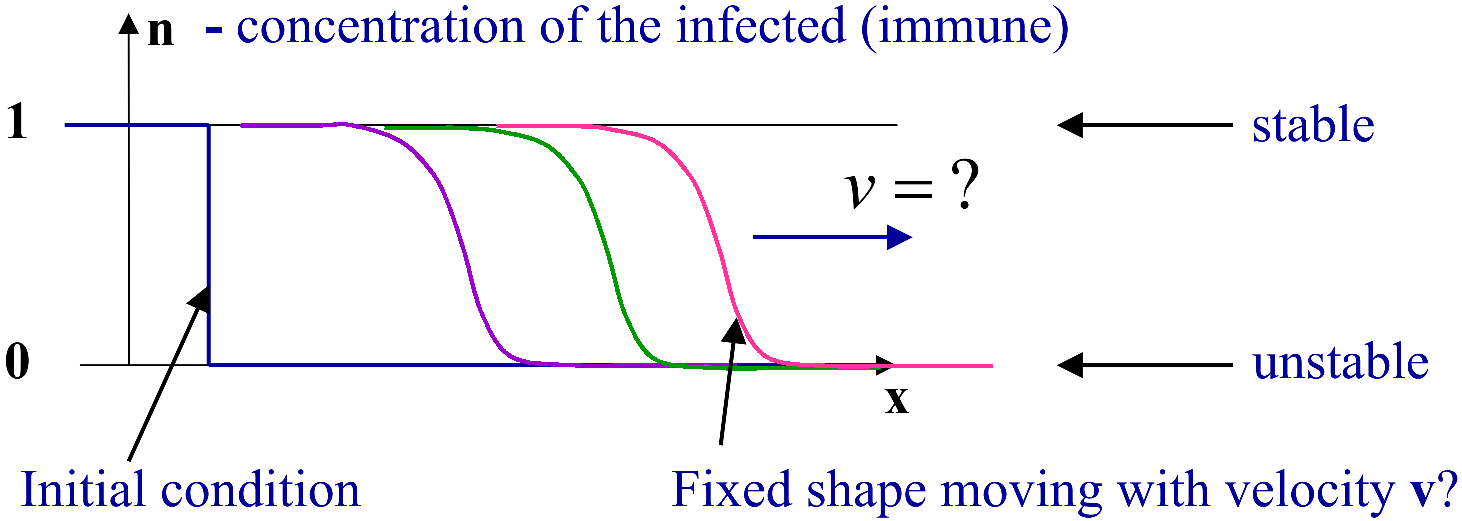




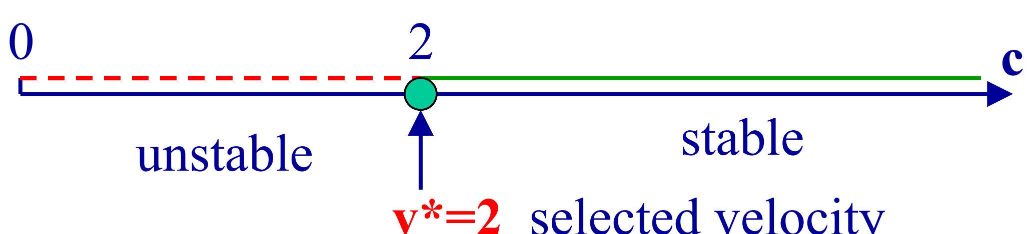


Infection front:
Fisher-Kolmogorov-Piskunov equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + n - n^2$$

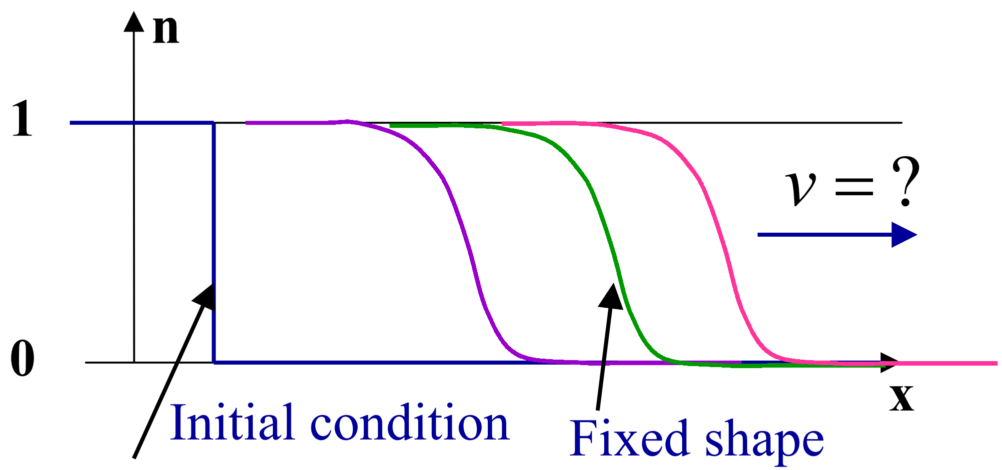


Solution exists for arbitrary **v** $\longrightarrow$ $n(x,t) = \Phi(x - vt)$



FKP equation: Instability at small velocities

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + n - n^2$$



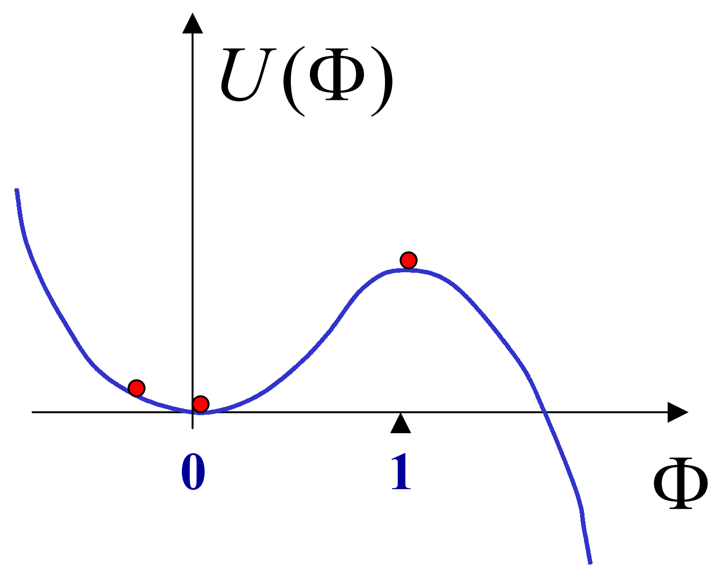
Solution exists for arbitrary **v**

$$n(x,t) = \Phi(x - vt)$$

$$\ddot{\Phi} = -v\dot{\Phi} - \frac{d}{d\Phi}\left(\frac{1}{2}\Phi^2 - \frac{1}{3}\Phi^3\right)$$

v - friction

$U(\Phi)$



v (friction) small → oscillations around 0

FKP equation: Leading edge analysis

$$\partial_t n = \partial_x^2 n + n - n^2$$

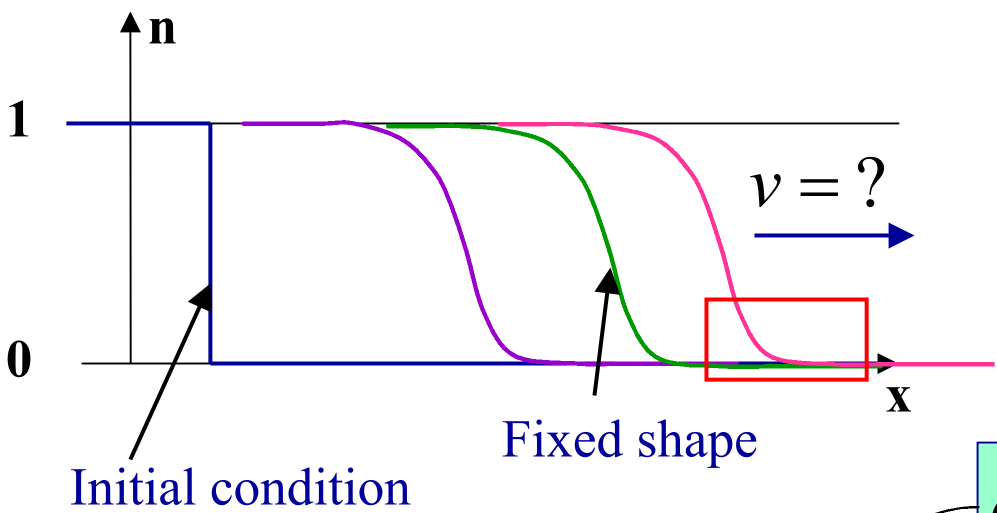
linearization

$$\partial_t n \approx \partial_x^2 n + n$$

Fourier transform

$$\partial_t n_k \approx (-k^2 + 1) n_k = \omega_k n_k$$

Assumption: Leading edge determines the velocity
(the front is pulled by the leading edge)



$$n(x,t) \approx \int_{-\infty}^{\infty} dk\, \tilde{n}_k\, e^{ikx + \omega_k t}$$

$$x = vt, \quad t \to \infty$$

$$n(vt,t) \approx \int_{-\infty}^{\infty} dk\, \tilde{n}_k\, e^{(ikv + \omega_k)t}$$

looking for stationary point in the moving frame

FKP equation: Leading edge analysis II

$$\partial_t n = \partial_x^2 n + n - n^2$$

Looking for stationary point in the moving frame

$$n(vt,t) \approx \int_{-\infty}^{\infty} dk\, \tilde{n}_k\, e^{(ikv+\omega_k)t}$$

~const

Stationary phase

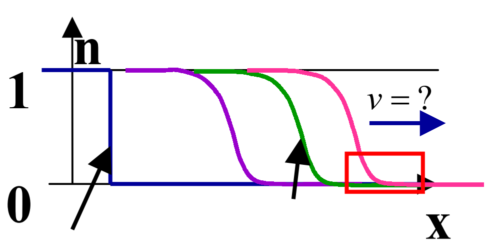
$$iv + \frac{d\omega_k}{dk} = 0$$

$$k^* = k^*(v)$$

$$n(vt,t) \approx e^{(ik^* v + \omega_{k^*})t} A$$

Stationarity in the moving frame:

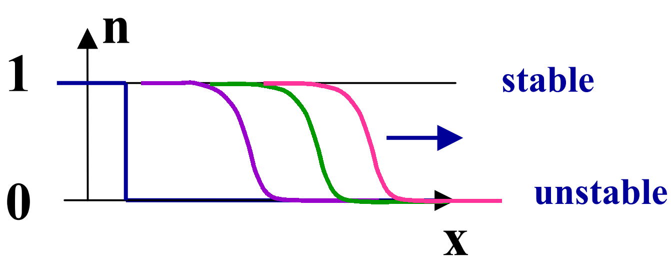
$$ik^* v + \omega_{k^*} = 0$$

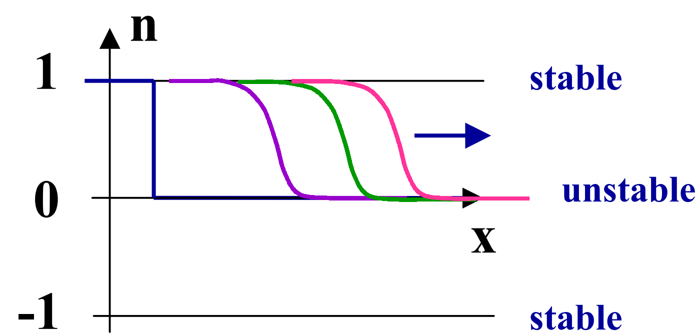


Selected velocity:

$$v^* = 2$$

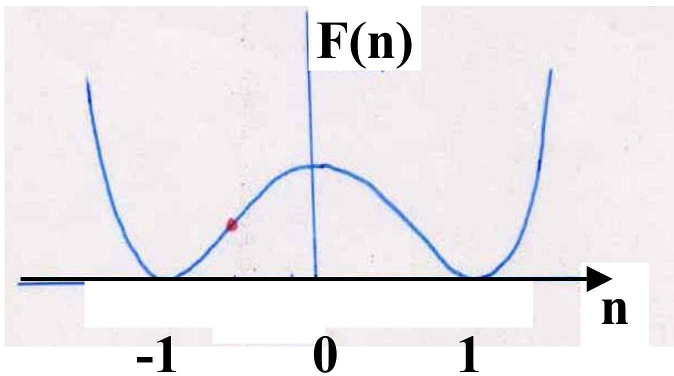
Conservation laws and moving fronts: Cahn-Hilliard equation

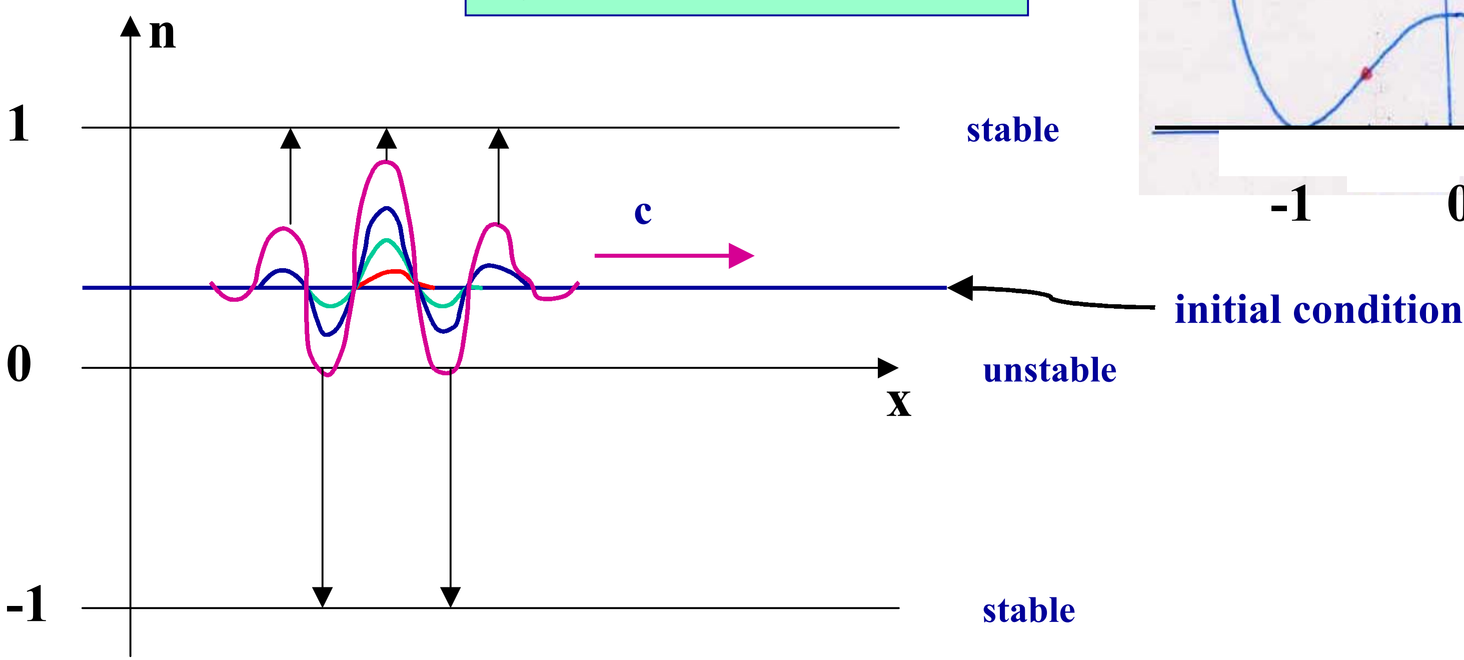




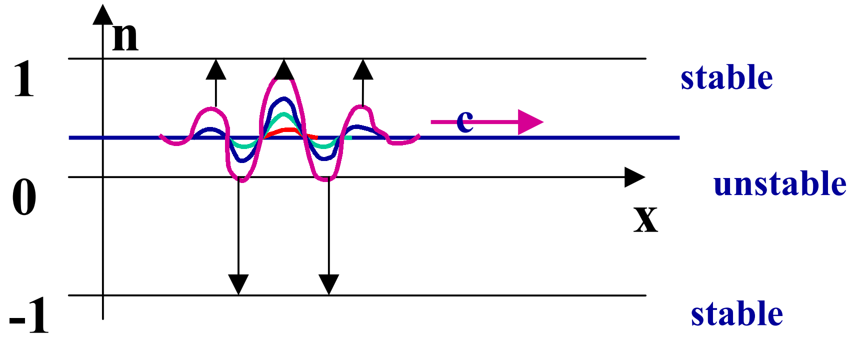
n is conserved:

$$\int_{-\infty}^{\infty} n(x,t)\,dx = const.$$





Conservation laws and moving fronts: Cahn-Hilliard equation



$$\frac{\partial n}{\partial t} = -\frac{\partial^2}{\partial x^2}\left(n - n^3 + \frac{\partial^2 n}{\partial x^2}\right)$$