Pattern Formation

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Introduction

(1) Why is there something instead of nothing?

Homogeneous vs. inhomogeneous systems
Deterministic vs. probabilistic description
Instabilities and symmetry breakings in homogeneous systems

(2) Can we hope to describe the myriads of patterns?

Notion of universality near a critical instability.
Common features of emerging patterns.
Example: Benard instability and visual hallucinations.
Notion of effective long-range interactions far from equilibrium.
Scale-invariant structures.

(3) Should we use macroscopic or microscopic equations?

Relevant and irrelevant fields -- effects of noise.
Arguments for the macroscopic.
Example: Snowflakes and their growth.
Remanence of the microscopic: Anisotropy and singular perturbations.

Patterns from stability analysis

(1) Local and global approaches.

Problem of relative stability in far from equilibrium systems.

(2) Linear stability analysis.

Stationary (fixed) points of differential equations.
Behavior of solutions near fixed points: stability matrix and eigenvalues.
Example: Two dimensional phase space structures
Lotka-Volterra equations, story of tuberculosis
Breaking of time-translational symmetry: hard-mode instabilities
Example: Hopf bifurcation: Van der Pole oscillator
Soft-mode instabilities: Emergence of spatial structures
Example: Chemical reactions - Brusselator.

(3) Critical slowing down and amplitude equations for the slow modes.

Landau-Ginzburg equation with real coefficients.
Symmetry considerations and linear combination of slow modes.
Boundary conditions - pattern selection by ramp.

(4) Weakly nonlinear analysis of the dynamics of patterns.

Secondary instabilities of spatial structures.
Eckhaus and zig-zag instability, time dependent structures.

(5) Complex Landau-Ginzburg equation

Convective and absolute instabilities of patterns.
Benjamin-Feir instability - spatio-temporal chaos.
One-dimensional coherent structures, noise sustained structures.

Patterns from moving fronts

(1) Importance of moving fronts: Patterns are manufactured in them.

Examples: Crystal growth, DLA, reaction fronts.
Dynamics of interfaces separating phases of different stability.
Classification of fronts: pushed and pulled.

(2) Invasion of an unstable state.

Velocity selection.
Example: Population dynamics.
Stationary point analysis of the Fisher-Kolmogorov equation.
Wavelength selection.
Example: Cahn-Hilliard equation and coarsening waves.

(3) Diffusive fronts.

Liesegang phenomena (precipitation patterns in the wake of diffusive reaction fronts - a problem of distinguishing the general and particular).

Literature

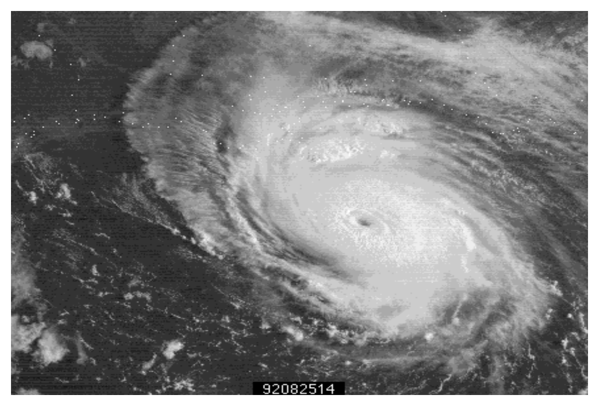
M. C. Cross and P. C. Hohenberg, **Pattern Formation Outside of Equilibrium**, Rev. Mod. Phys. 65, 851 (1993).

J. D. Murray, **Mathematical Biology**, (Springer, 1993; ISBN-0387-57204).

Why is there Something instead of Nothing? (Leibniz)

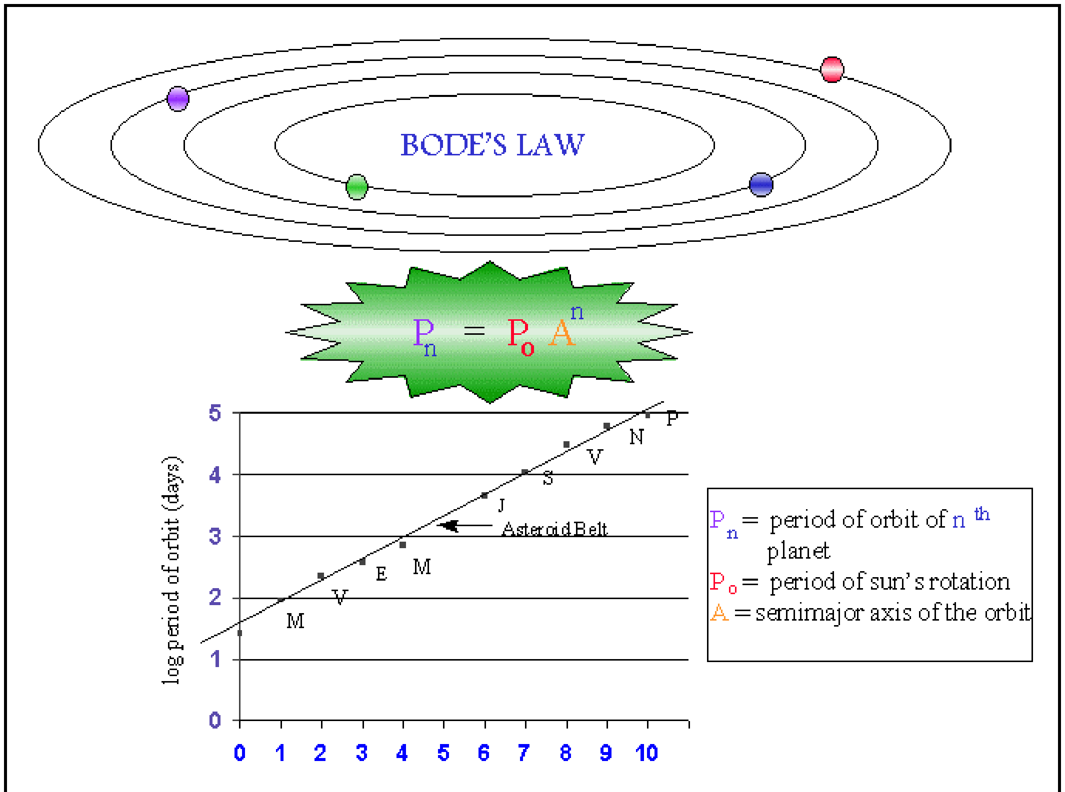
Homogeneous (amorphous) vs. inhomogeneous (structured)

Actors
and
spectators
(N. Bohr)



Deterministic vs. probabilistic aspects

The question of the origins of order:



(Cornell Universty)

Titius-Bode law

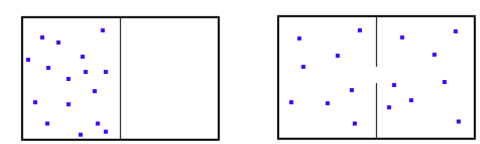
Bishop to Newton:

Now that you discovered the laws governing the motion of the planets, can you also explain the regularity of their distances from the Sun?

Newton to Bishop:

I have nothing to do with this problem.
The initial conditions were set by God.
?

Thermo:
S=max



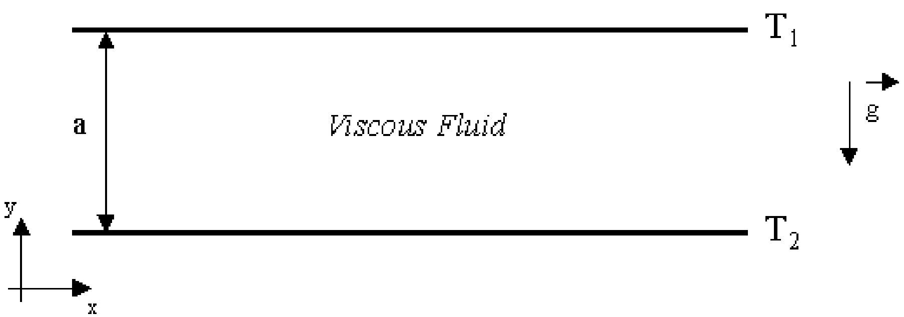
Equilibrium is independent of initial conditions (at given constraints)

Stability

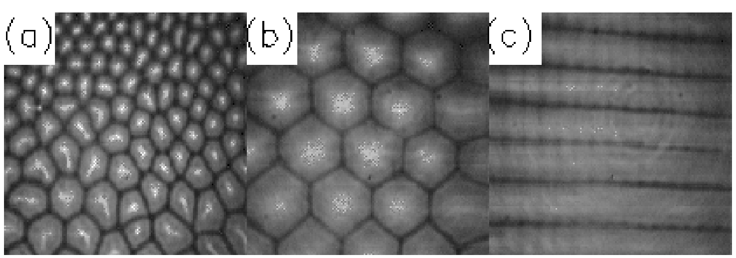
Instabilities and Symmetry Breakings

Basic approach: Understand more complex through studies of (symmetry breaking) instabilities of less complex

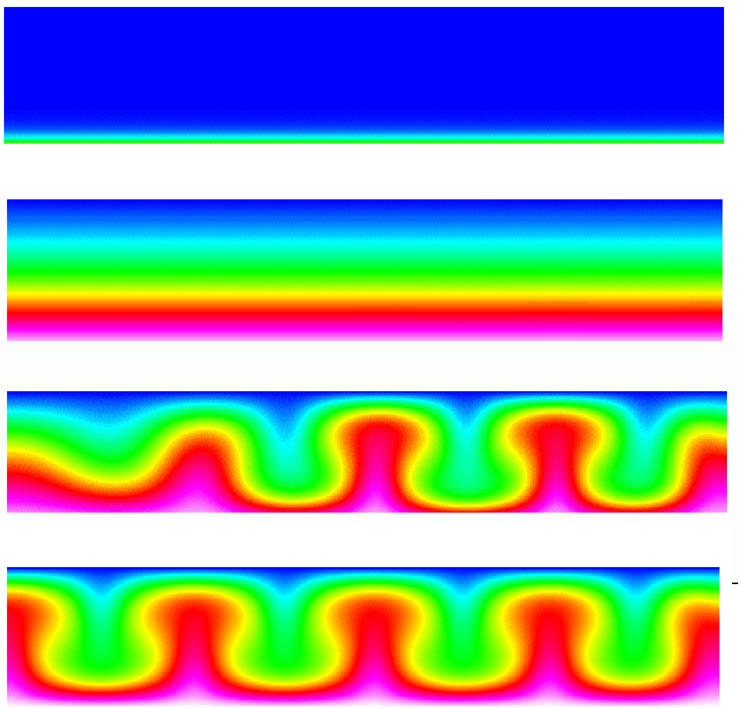
Rayleigh-Bénard:



M.Schatz (shadowgraph images of convection patterns):

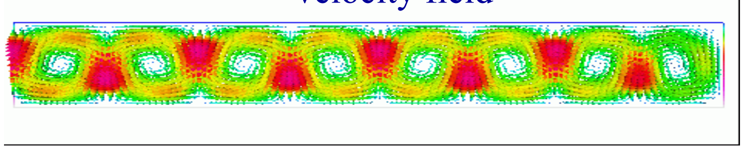


temperature field



velocity field

(Elmer Co.)

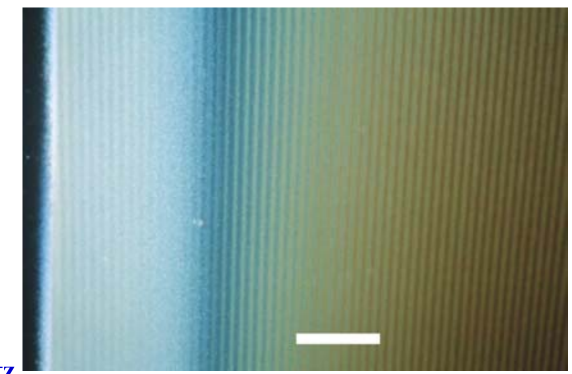


The wonderful world of stripes

Clouds

Characteristic length:
$\sim 10^{2}$ m

P. Hantz



Precipitation patterns in gels

$CuCl_2 + NaOH \rightarrow$

<u>CuO</u> + ...

$\sim 10^{-4}$ m

The massive white dunes of Sand Mountain, southeast of Fallon, Nevada. This is one of the few "booming dunes"

NASA

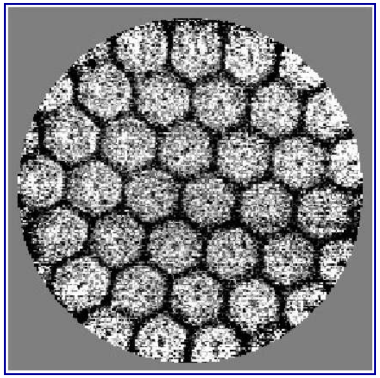
Sand dunes

$\sim 10^{-1}$ - 10^{4} m

Visual Hallucinations and the Bénard Instability

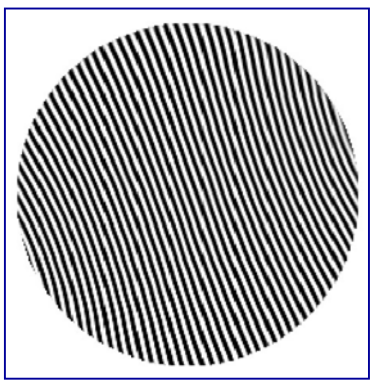
Bénard experiments (G. Ahlers et al.)

Visual hallucinations (H. Kluver)

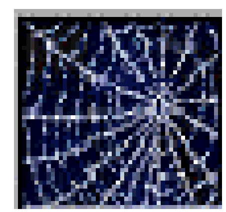


Caleidoscope

(lattice, network, grating honeycomb)

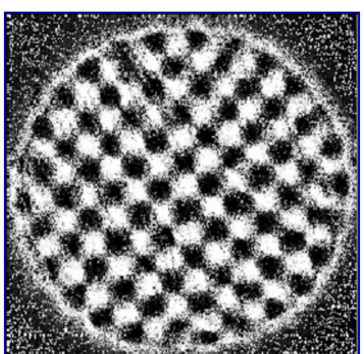


tunnel

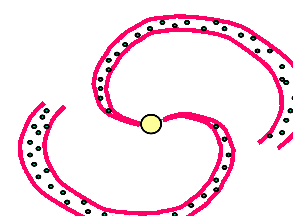


funnel

spiral

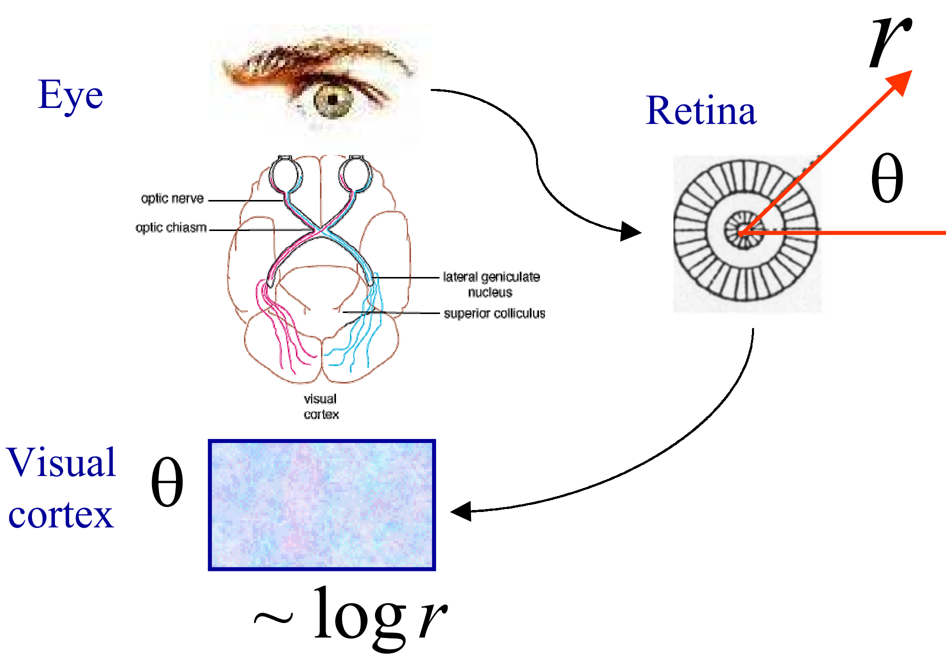


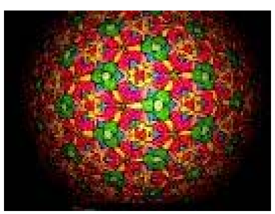
cobweb

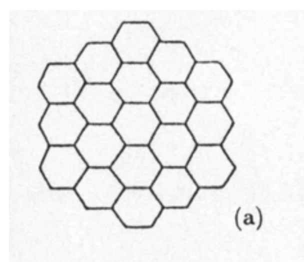


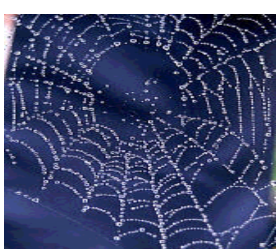
Visual hallucinations: retina ↔ visual cortex mapping

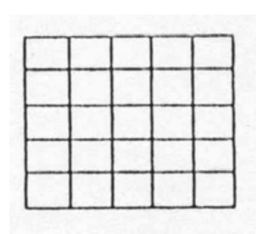
J. D. Cowan



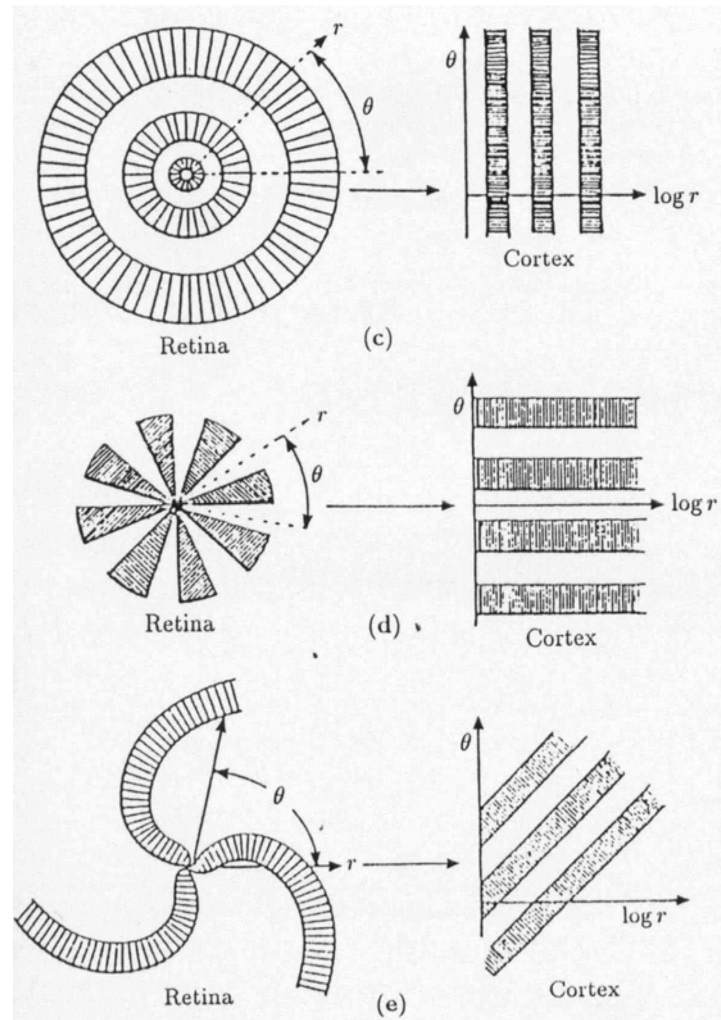








Retina Visual cortex

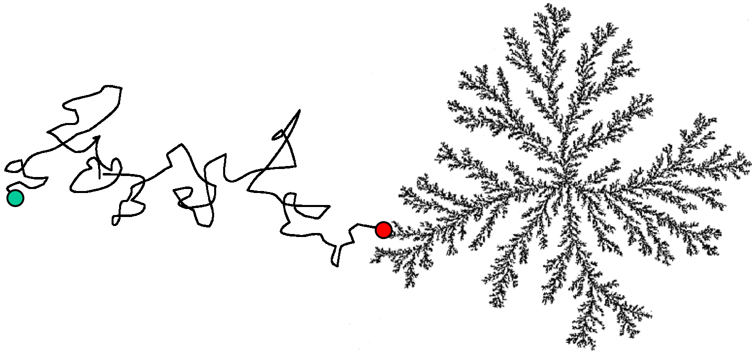


Scale Invariant Structures

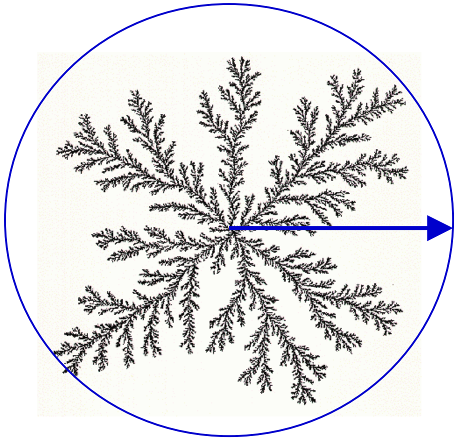
Oak tree

MgO_2 in Limestone C.-H. Lam

DLA (diffusion limited aggregation)



1 million particles



N=100 million (H. Kaufman)

$$R \approx N^{1/D}$$

Level of description: Microscopic or macroscopic?

(1) No two snowflakes are alike

(2) All six branches are alike

→ Parameters determining growth fluctuate on lengthscales larger than 1*mm*.

(3) Sixfold symmetry

→ Microscopic structure is relevant on macroscopic scale.

(4) Twelvefold symmetry (not very often)

→ Initial conditions may be remembered.

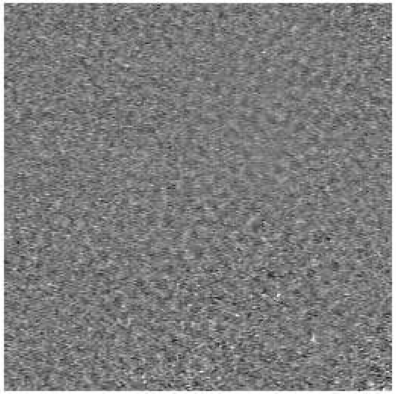
Fluctuations and Noise

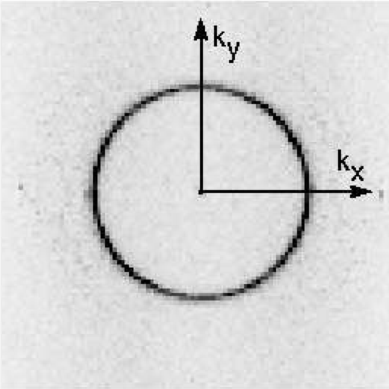
disorder
homogeneous

instability
large fluctuations

order

Rayleigh-Benard near but below the convection instability:





G. Ahlers et al.

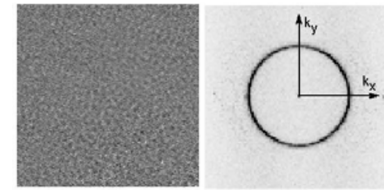
Power spectrum

$$S(k) \sim \langle |\, \rho_k \,|^2 \rangle$$

$$\rho_k = \frac{1}{V} \sum_x e^{ikx} \rho(x)$$

Emergence of spatial structures: Soft mode instabilities



$$n \rightarrow n(t) \rightarrow n(x,t)$$

$$\dot{n} = f(n, \partial_x n, \partial_x^2 n, \ldots, \lambda)$$

Spatial mixing: convection, diffusion, … → Reaction-diffusion systems

$$\dot{n}_1 = D_1 \Delta n_1 + f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = D_2 \Delta n_2 + f_2(n_1, n_2, \lambda)$$

Chemical reactions in gels
(model example: Brussellator)

Stability analysis:

(1) Stationary homogeneous solutions:

$$f_1(n_1^*, n_2^*, \lambda) = 0$$
$$f_2(n_1^*, n_2^*, \lambda) = 0$$

Stab

Emergence of structures: Stability analysis

$$\dot{\mathbf{n}} = D\Delta\mathbf{n} + \mathbf{f}(\mathbf{n},\lambda)$$

$$\mathbf{n}^*$$

Linearization

$$\mathbf{n} = \mathbf{n}^* + \delta\mathbf{n}_k e^{ikx}$$

$$\delta\dot{\mathbf{n}}_k = \mathbf{A}_\lambda(k)\,\delta\mathbf{n}_k$$

Stability matrix

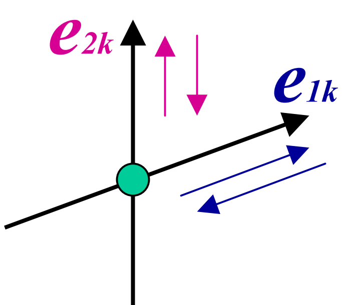
$$\tilde{\mathbf{A}}_\lambda \Rightarrow \begin{pmatrix} \omega_{1,\lambda}(k) & & \\ & \ddots & \\ & & \omega_{n,\lambda}(k) \end{pmatrix}$$

Eigenvalues

Solution

$$\delta\mathbf{n}_k = \sum_{j=1}^{n} c_{jk}\,\mathbf{e}_{jk}\, e^{\omega_{j\lambda}(k)t}$$

Eigenvectors



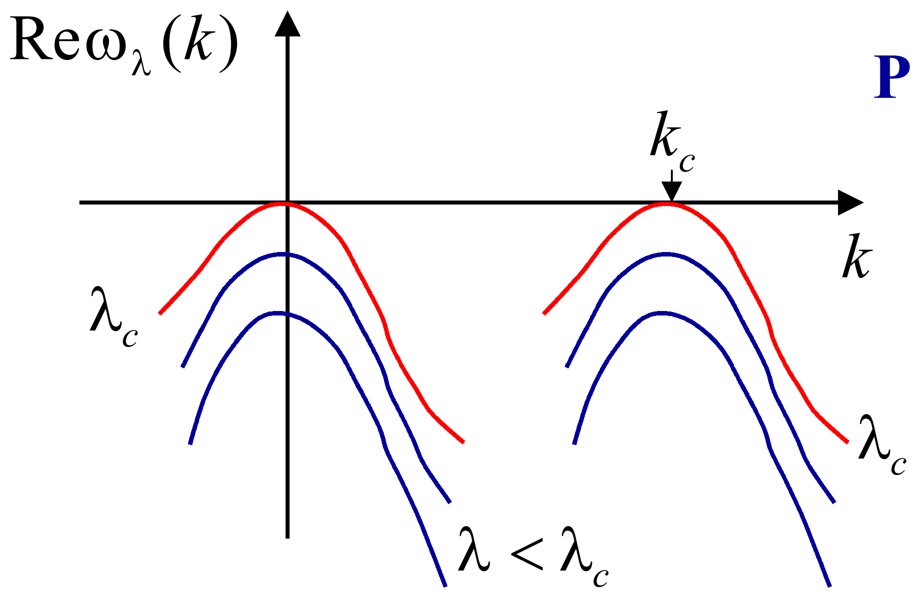
Critical slowing down and classification of instabilities

$$\omega_{\lambda 1,2}(k) \Rightarrow \omega_{\lambda}(k)$$

- with the largest real part

Instability: $\mathrm{Re}\,\omega_{\lambda}(k) \to 0^{-}$

$$\lambda \to \lambda_c$$

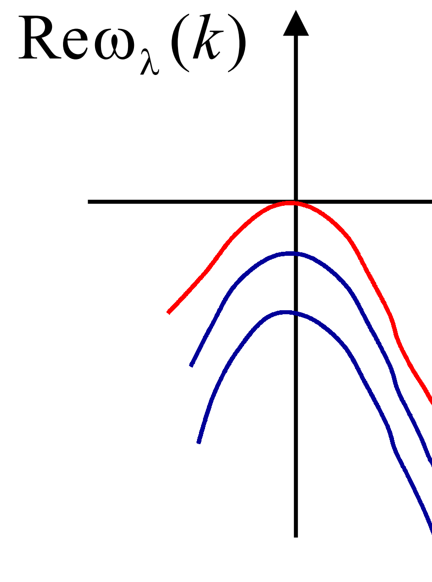


Possibilities:

$$k_c \begin{cases} = 0 \\ \neq 0 \end{cases}$$

$$\mathrm{Im}\,\omega_{\lambda_c}(k_c) \begin{cases} = 0 & \text{soft} \\ \neq 0 & \text{hard} \end{cases}$$

Classification of instabilities - emerging structures



$k_c = 0$

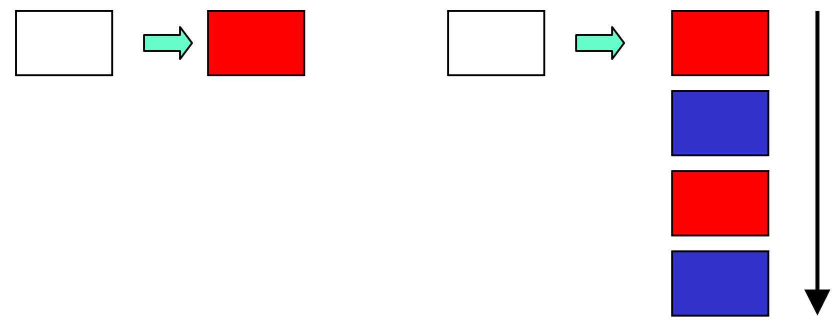
$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$

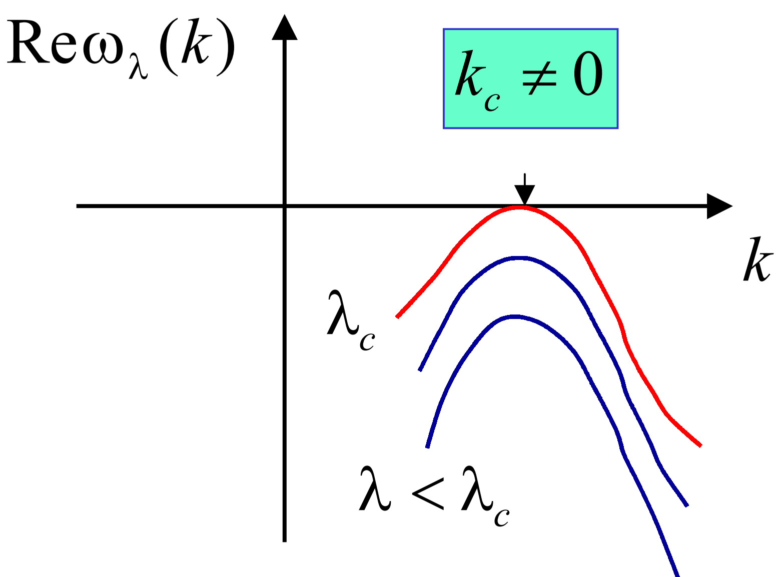
$\mathrm{Im}\,\omega_{\lambda_c}(k_c) \neq 0$

spatially homogeneous

stationary

limit cycle



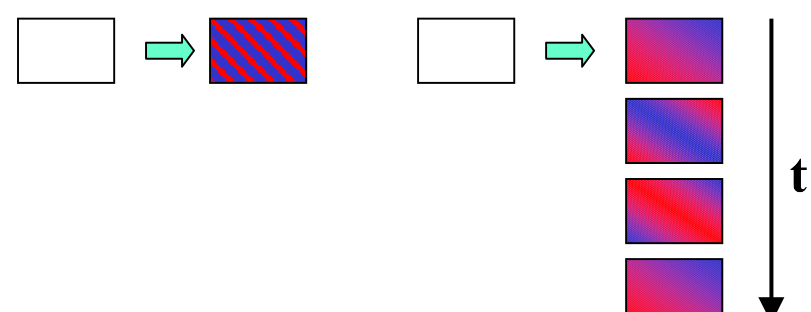


$k_c \neq 0$

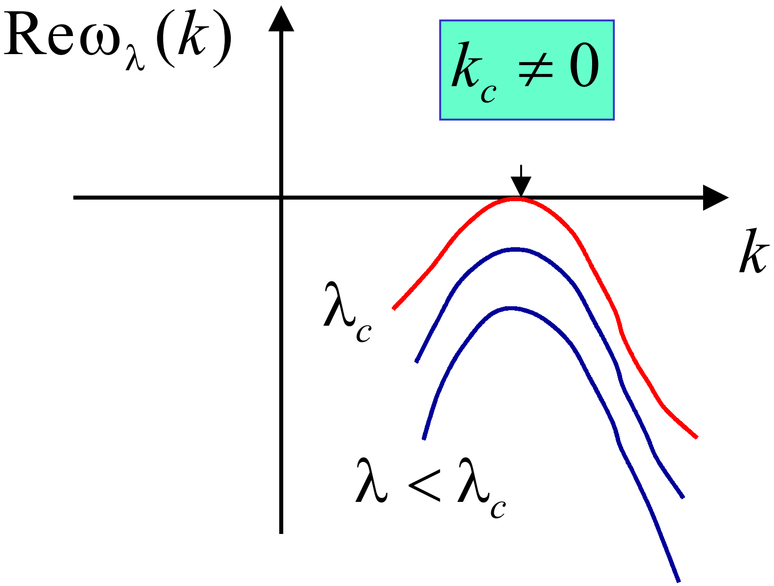
spatially structured

stationary

time- and space-dependent

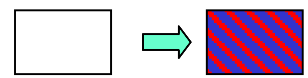


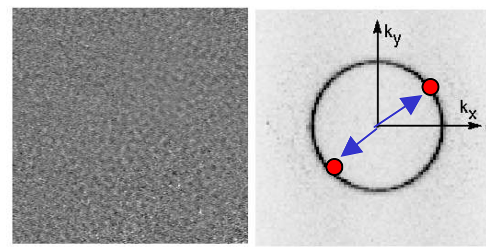
Stationary structures emerging in d=2 homogeneous systems



$$k_c \neq 0$$

$$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$$

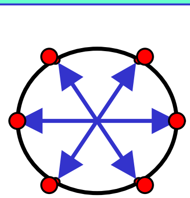


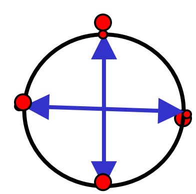


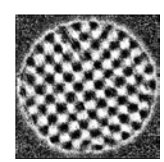
d=2
isotropy

$$\omega_\lambda(\vec{k}) \Rightarrow \omega_\lambda(k)$$

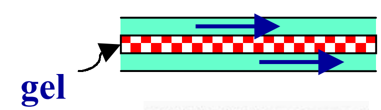
$$\delta n_{\vec{k}} \sim a_{\vec{k}} e^{i\vec{k}\vec{x}+\omega_\lambda(k)t}$$

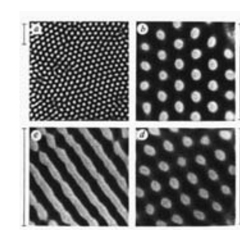


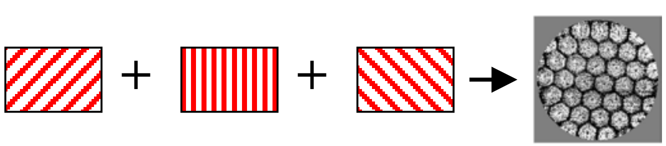


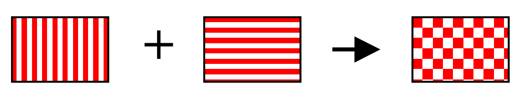


Swinney et al. 1991
Turing patterns

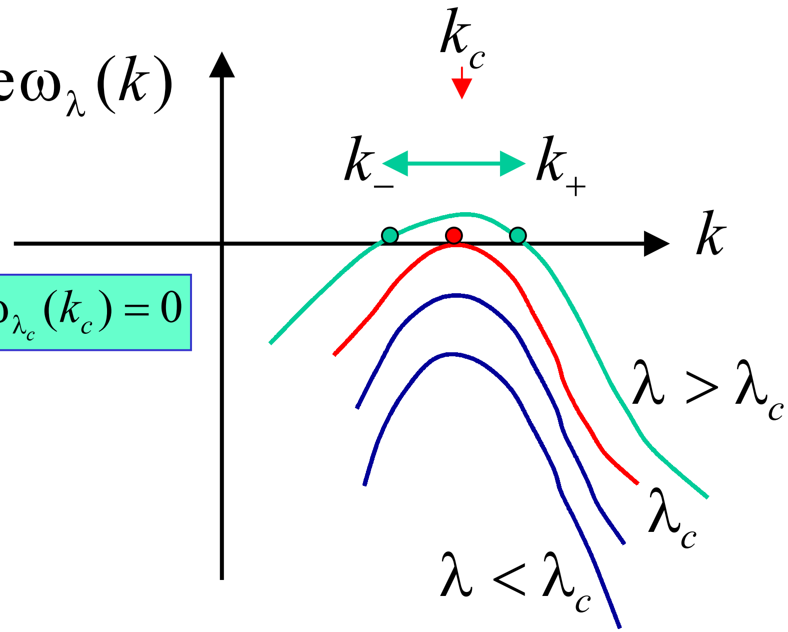








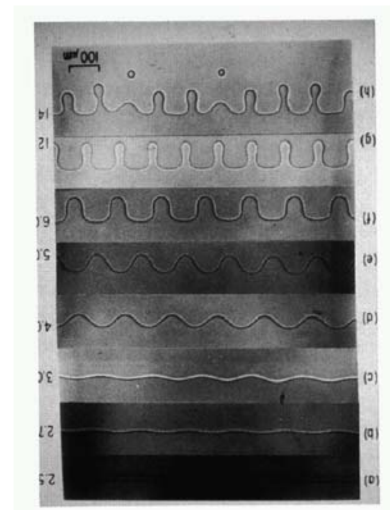
Beyond the instability: Amplitude equation for slow modes



$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$

$\lambda > \lambda_c$ Band of unstable modes

What is the steady state?



$\lambda >> \lambda_c$

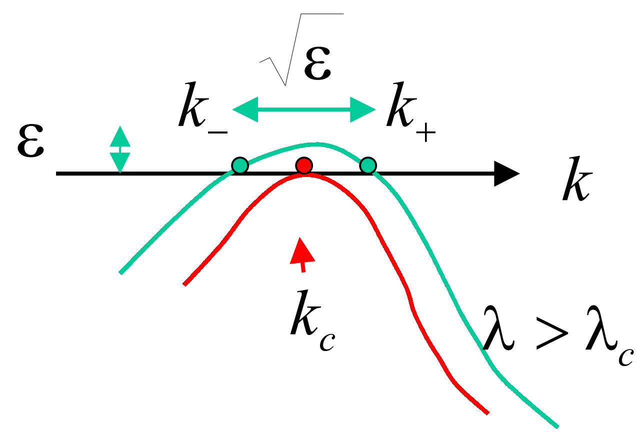
$\lambda \geq \lambda_c$

$\lambda = \lambda_c$

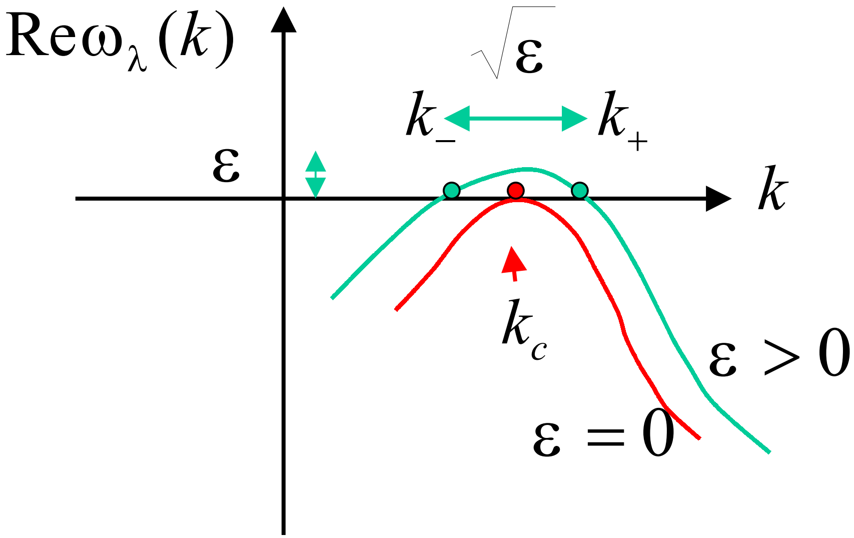
$\mathrm{Re}\,\omega_\lambda(k)$ smooth function of

$$\omega_\lambda(k) \approx \underbrace{\lambda - \lambda_c}_{\varepsilon} - a(k - k_c)^2$$

ε **control parameter from now on**



Amplitude equation: Characteristic lengths and times



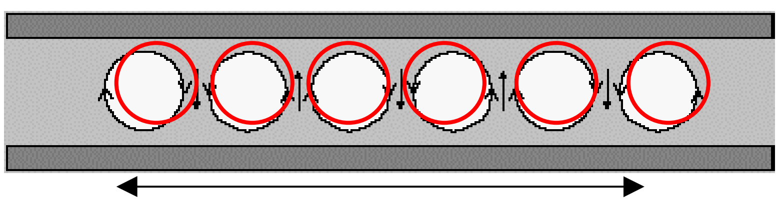
$\varepsilon > 0$ Band of unstable modes

$$\omega_\lambda(k) \approx \varepsilon - a(k - k_c)^2$$

$$n(x,t) - n^* = \int_{-\infty}^{\infty} dk\, \tilde{n}_k\, e^{ikx + \omega_\lambda(k)t}$$

$$\approx e^{ik_c x} \underbrace{\int_{k_-}^{k_+}}_{\sim\sqrt{\varepsilon}} dk\, \tilde{n}_k\, e^{\overbrace{i(k-k_c)x}^{\sim\sqrt{\varepsilon}} + \overbrace{\omega_\lambda(k)t}^{\sim\varepsilon}}$$

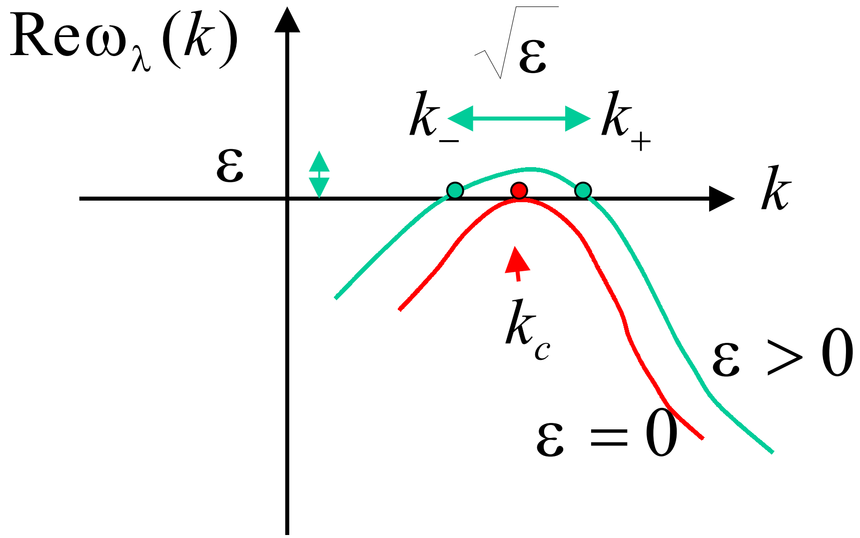
$$\approx e^{ik_c x} \sqrt{\varepsilon}\, A_0(\sqrt{\varepsilon}\, x, \varepsilon\, t)$$



$\xi \sim 1/\sqrt{\varepsilon}$

Variation of the amplitude of the periodic structure on lenghtscale $\xi \sim 1/\sqrt{\varepsilon}$ **and on timescale** $\tau \sim 1/\varepsilon$.

Amplitude equation



$\varepsilon > 0$ Band of unstable modes

$$\omega_\lambda(k) \approx \varepsilon - a(k - k_c)^2$$

$$\xi \sim 1/\sqrt{\varepsilon} \qquad \tau \sim 1/\varepsilon$$

$$n(x,t) - n^* \approx e^{ik_c x}\sqrt{\varepsilon}\,A_0(\sqrt{\varepsilon}\,x, \varepsilon\,t) \equiv e^{ik_c x} A(x,t)$$

Plug it in the original equation and expand.

Amplitude equation:

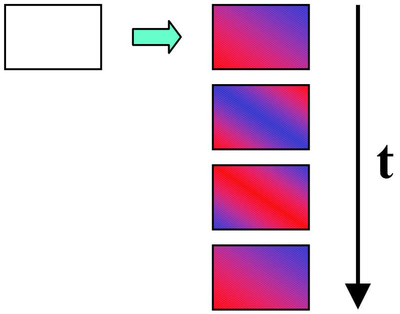
$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

Complex Landau-Ginzburg equation

$$k_c \neq 0, \quad \mathrm{Im}\,\omega_{\lambda_c}(k_c) \neq 0$$

$$n(x,t) - n^* \approx e^{i[k_c x + \mathrm{Im}\,\omega(k_c)t]} A(x,t)$$

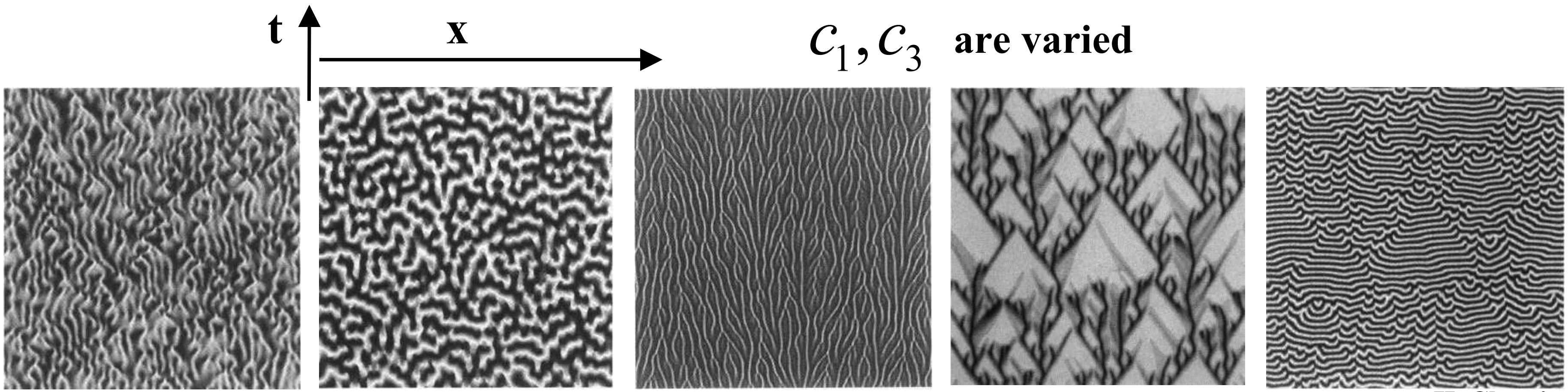
time- and space-dependent



$$v = \mathrm{Im}\,\omega(k_c)/k_c$$

$$\frac{\partial A}{\partial t} + v\frac{\partial A}{\partial x} = \varepsilon A + (1+ic_1)\frac{\partial^2 A}{\partial x^2} - (1-ic_3)|A|^2 A$$

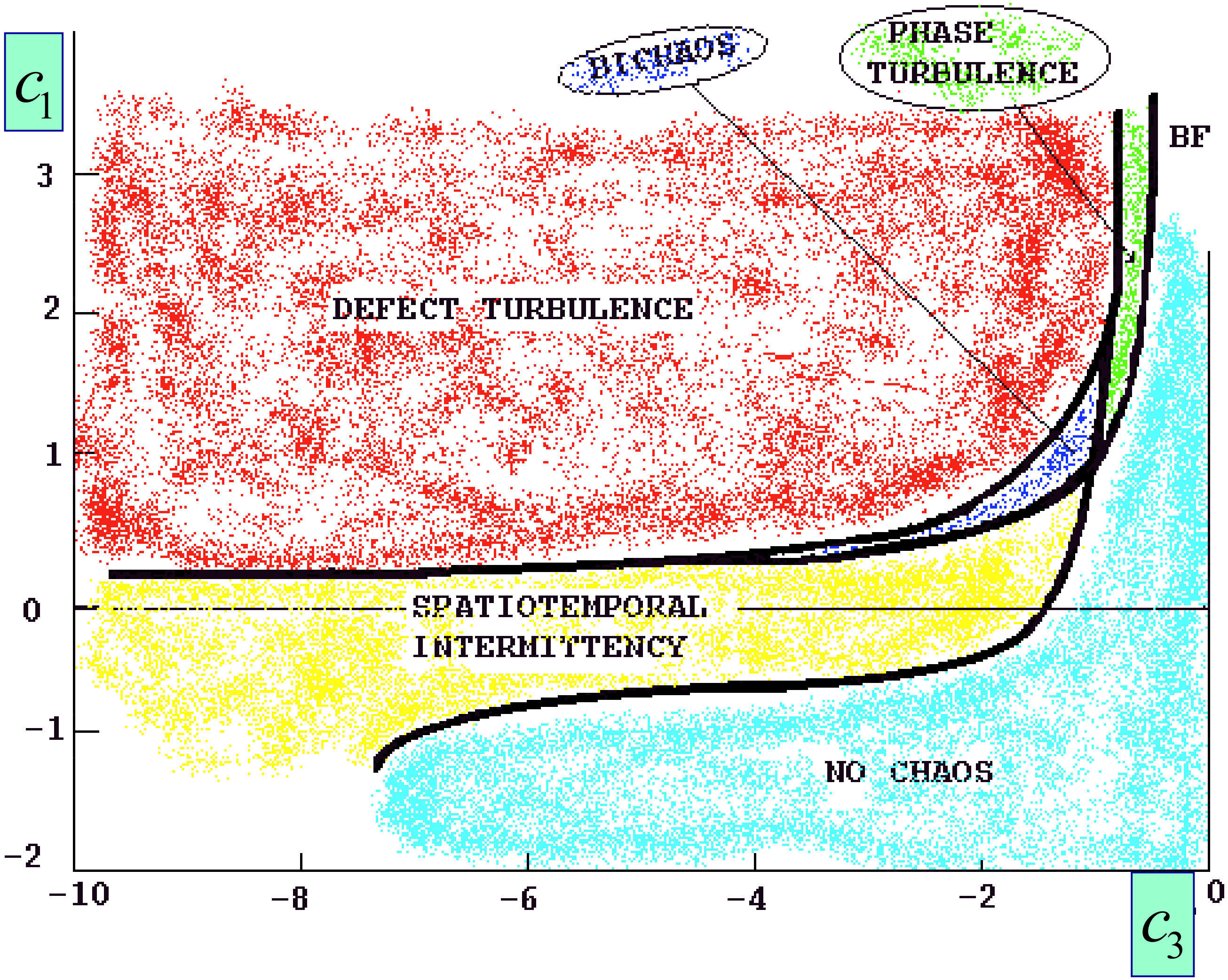
Velocity of the wave



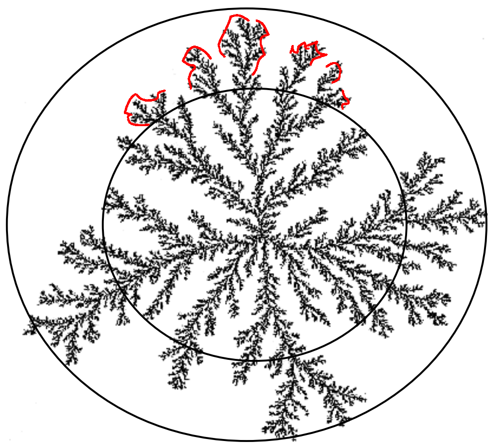
H. Chate'

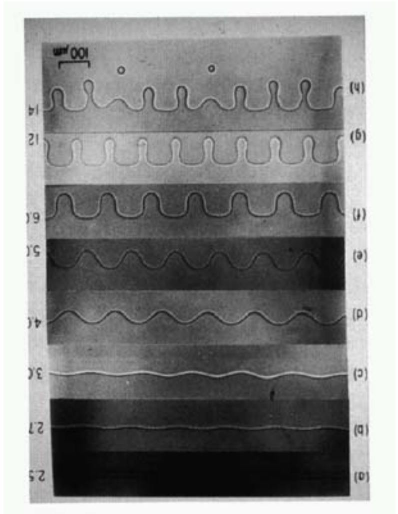
CLGE - Phase diagram

$$\frac{\partial A}{\partial t} = \varepsilon A + (1 + ic_1)\frac{\partial^2 A}{\partial x^2} - (1 - ic_3)|A|^2 A$$



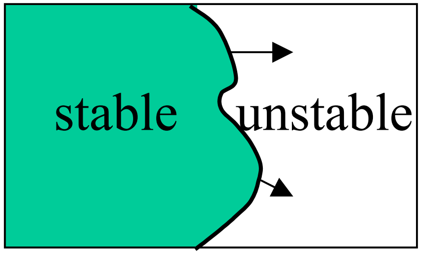
Fronts separating stable and unstable phases





crystallization fronts
chemical reaction fronts

The problems:



(1) What is the speed of the front?

(2) Is there any nontrivial structure
in the wake of the front?