Derivation of the amplitude equation for the Swift-Hohenberg model

Z. Rácz
Institute for Theoretical Physics–HAS, Eötvös University, 1117 Budapest, Hungary

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Arguments presented in the lectures suggest that the form of the amplitude equation is rather general. Here we show by detailed analysis that, near its instability point, the Swift-Hohenberg model indeed yields the general form of the amplitude equation.

I. THE MODEL EQUATION

The Swift-Hohenberg model [1] is specially chosen to make the calculation of the amplitude equation as simple as possible (its origin is in hydrodynamics but establishing the link is not straightforward). The system is described by a scalar field $u(x,t)$ whose dynamics for the one dimensional case is given by the so called Swift-Hohenberg (SH) equation:

$$\partial_t u = \varepsilon u - (\partial_x^2 + k_c^2)^2 u - u^3 \,. \tag{1}$$

Here ε is the control parameter while, as shown by the linear stability analysis, k_c is the critical wave number.

II. LINEAR STABILITY ANALYSIS

It is easy to see that $u^* = 0$ is a homogeneous, stationary solution of the equation. Linearizing around this solution means that the u^3 term is dropped from the equation. The spatial Fourier transform of the linearized equation gives us the following solution for the Fourier components of $u(x,t) = \int e^{ikx} u_k(t) dk$

$$u_k(t) = u_k(0) e^{\omega_k t} \,, \tag{2}$$

where the characteristic frequencies determining the stability are given by

$$\omega_k = \varepsilon - (k^2 - k_c^2)^2 \,. \tag{3}$$

One can see that the system becomes unstable for $\varepsilon \geq 0$ and, indeed, the characteristic wavenumber of the emerging pattern at $\varepsilon_c = 0$ should be k_c.

III. PERTURBATIONS FOR SMALL ε

For small ε, as discussed in the main part of the lecture notes, the characteristic length-scales and time-scales are given by $1/\sqrt{\varepsilon}$ and $1/\varepsilon$, respectively. Thus we are seeking the solution of eq.(1) in the form

$$u(x,t) \approx e^{ik_c x} A(\varepsilon^{1/2} x, \varepsilon t) \tag{4}$$

or, more precisely, we make the following expansion in powers of $\varepsilon^{1/2}$ (remember that we showed that $A \sim \varepsilon^{1/2}$ in the main part of the notes)

$$\begin{aligned} u(x,t) = \; & \varepsilon^{1/2} A_0(X,T)\, \Phi(x) + & (5)\\ & \varepsilon \;\; B_0(X,T)\; \Psi(x) \; + & (6)\\ & \varepsilon^{3/2} \;\; C_0(X,T)\; \zeta(x) \,. & (7) \end{aligned}$$

where

$$X = \varepsilon^{1/2} x \quad \text{and} \quad T = \varepsilon t \,. \tag{8}$$

The task now is to substitute the above expansion into eq.(1) and collect the terms proportional to $\varepsilon^{1/2}$, ε, up to $\varepsilon^{3/2}$. It is not by chance that we expand up to $\varepsilon^{3/2}$, the amplitude equation emerges only at this order.

The calculations are somewhat simplified if we note that the terms in $u(x,t)$ consist of the product of two functions with one of them depending on x while the other one on X and T. This means that we can replace the derivatives by

$$\partial_t \to \varepsilon \partial_T \qquad ; \qquad \partial_x \to \partial_x + \varepsilon^{1/2} \partial_X \,. \tag{9}$$

IV. ORDER OF THE TERMS IN THE SH EQUATION

The left-hand side of eq.(1) is already of the order $\varepsilon^{3/2}$:

$$\partial_t u \;\to\; \varepsilon^{3/2} \partial_T A_0 \, \Phi \,. \tag{10}$$

The same is true about the 1st and 3rd terms on the right-hand side

$$\varepsilon u \;\to\; \varepsilon^{3/2} A_0 \, \Phi \quad , \quad -u^3 \;\to\; -\varepsilon^{3/2} (A_0 \, \Phi)^3 \tag{11}$$

The complicated piece comes from the spatial derivatives.

$$-(\partial_x^2 + k_c^2)^2 u = -\left[(\partial_x + \varepsilon^{1/2} \partial_X)^2 + k_c^2 \right]^2 u \,. \tag{12}$$

Since u is at least of the order $\varepsilon^{1/2}$, the bracket $[]^2$ must be expanded up to order ε:

$$\begin{aligned} []^2 \;&=\; \left[\partial_x^2 + k_c^2 + 2\varepsilon^{1/2} \partial_x \partial_X + \varepsilon \partial_X^2 \right]^2 & (13)\\ &=\; \left[\mathcal{L} + 2\varepsilon^{1/2} \partial_x \partial_X + \varepsilon \partial_X^2 \right]^2 & (14)\\ &=\; \mathcal{L}^2 + 4\varepsilon^{1/2} \mathcal{L} \partial_x \partial_X + 4\varepsilon \partial_x^2 \partial_X^2 + 2\varepsilon \mathcal{L} \partial_X^2 & (15) \end{aligned}$$

where we introduced the operator

$$\mathcal{L} = \partial_x^2 + k_c^2 \,. \tag{16}$$

The $[]^2 u$ term produces various orders in $\varepsilon^{1/2}$ and, replacing u by its expansion, we obtain

$$[]^2 u = \varepsilon^{1/2} A_0 \mathcal{L}^2 \Phi + \varepsilon \left[4\partial_X A_0 \mathcal{L} \partial_x \Phi + B_0 \mathcal{L}^2 \Psi\right] + \tag{17}$$
$$\varepsilon^{3/2} \left[4\partial_X^2 A_0 \partial_x^2 \Phi + 2\partial_X^2 A_0 \mathcal{L} \Phi + 4\partial_X B_0 \mathcal{L} \partial_x \Psi + C_0 \mathcal{L}^2 \zeta\right]$$

Collecting now the terms from eqs.(10,11,17), we can analyse them order by order of the power of $\varepsilon^{1/2}$.

V. ORDERS $\varepsilon^{1/2}$ AND ε

The order $\varepsilon^{1/2}$ equation is simple

$$A_0 \mathcal{L}^2 \Phi = A_0 (\partial_x^2 + k_c^2)^2 \Phi = 0 \,. \tag{18}$$

Its periodic solution, taking into account that $A_0 \Phi$ should be real, is given by

$$A_0 \Phi = \tilde{A}_0 e^{ik_c x} + \tilde{A}_0^* e^{-ik_c x} \,. \tag{19}$$

Note that $\tilde{A}_0$ is not determined at this order, it is just an integration constant in solving eq.(18).

At order ε, one finds

$$4\partial_X A_0 \mathcal{L} \partial_x \Phi + B_0 \mathcal{L}^2 \Psi = 0 \,. \tag{20}$$

Since $\mathcal{L}\Phi = 0$, the above equation reduces to

$$B_0 \mathcal{L}^2 \Psi = 0 \tag{21}$$

and so, to order ε, the SH equation is solved by

$$B_0 \Psi = \tilde{B}_0 e^{ik_c x} + \tilde{B}_0^* e^{-ik_c x} \,. \tag{22}$$

where $\tilde{B}_0$ is again an integration constant undetermined at this stage.

VI. ORDER $\varepsilon^{3/2}$

This is the first nontrivial order. Collecting the $\varepsilon^{3/2}$ terms from eqs.(10,11,17), we find

$$\begin{aligned} \Phi \partial_T A_0 &= \Phi A_0 - (\Phi A_0)^3 - 4\partial_X^2 A_0 \partial_x^2 \Phi \\ &\quad - \underline{2\partial_X^2 A_0 \mathcal{L}\Phi} - \underline{4\partial_X B_0 \mathcal{L} \partial_x \Psi} - C_0 \mathcal{L}^2 \zeta \end{aligned} \tag{23}$$

Since according to eqs.(18) and (21), we have $\mathcal{L}\Phi = 0$ and $\mathcal{L}\Psi = 0$, the underlined terms in eq.(23) disappear.

It follows from eq.(19) that $\partial_x^2 \Phi = -k_c^2 \Phi$ and thus, the 3rd term on the right hand side eq.(23) simplifies to

$$-4\partial_X^2 A_0 \partial_x^2 \Phi = 4k_c^2 \Phi \partial_X^2 A_0 \,. \tag{24}$$

Furthermore, the cubic term can be written as

$$\begin{aligned} (\Phi\, A_0)^3 &= e^{3ikx} \tilde{A}_0^3 + e^{-3ikx} \tilde{A}_0^{*\,3} + \\ &\quad 3\,(e^{ikx} \tilde{A}_0 + e^{-ikx} \tilde{A}_0^*)\, |\tilde{A}_0|^2 \,. \end{aligned} \tag{25}$$

Note now that all the terms examined so far were proportional either to $e^{ik_c x}$ or $e^{-ik_c x}$. The cubic term, however, yield terms $\sim e^{\pm 3ik_c x}$ which must be cancelled by the only term left, $C_0 \mathcal{L}^2 \zeta$. But the $C_0 \mathcal{L}^2 \zeta$ term cannot contain terms $\sim e^{\pm ik_c x}$. Indeed, if ζ would contain $e^{\pm ik_c x}$ terms then they would be annihilated by the operator $\mathcal{L}$.

Thus collecting separately the terms $\sim e^{\pm ik_c x}$ and $\sim e^{\pm 3ik_c x}$ in eq.(23), one finds the following two equations

$$\Phi \partial_T A_0 = \Phi A_0 - 3\,|\tilde{A}_0|^2 \Phi A_0 + 4k_c^2 \Phi \partial_X^2 A_0 \tag{26}$$

$$0 = -e^{3ikx} \tilde{A}_0^3 + e^{-3ikx} \tilde{A}_0^{*\,3} - C_0 \mathcal{L}^2 \zeta \,. \tag{27}$$

Eq.(27) determines ζ and, since it gives an $\varepsilon^{3/2}$ contribution to the amplitude, we shall not consider it any more. Eq.(26) can be written as two equations by separating the terms $\sim e^{\pm ik_c x}$. These equations are each other's complex conjugates, thus it is sufficient to write out one of them

$$\partial_T \tilde{A}_0 = \tilde{A}_0 - 3\,|\tilde{A}_0|^2\, \tilde{A}_0 + 4k_c^2\, \partial_X^2 \tilde{A}_0 \,. \tag{28}$$

Returning to variables $T = t\varepsilon$ and $X = x\varepsilon^{1/2}$, one finds

$$\partial_t \tilde{A}_0 = \varepsilon \tilde{A}_0 - 3\varepsilon\, |\tilde{A}_0|^2\, \tilde{A}_0 + 4k_c^2\, \partial_X^2 \tilde{A}_0 \,. \tag{29}$$

Making additional scale-changes in x and absorbing the $\varepsilon^{1/2}$ scale into $\tilde{A}_0$:

$$x \to 2k_c x \qquad ; \qquad A = (3\varepsilon)^{1/2} \tilde{A}_0 \,, \tag{30}$$

we arrive at the standard from of the amplitude equation

$$\partial_t A = \varepsilon A - |A|^2\, A + \partial_x^2 A \,. \tag{31}$$

The analysis of this equation can be found in the main line of the lecture notes.

[1] M. C. Cross and P. C. Hohenberg, *Pattern formation outside equilibrium*, Rev. Mod. Phys. **65**, 851-1112 (1993).