Casimir effect in the nonequilibrium steady state of a quantum spin chain

D. L. González-Cabrera,[1,3] Z. Rácz,[2] and F. van Wijland[3]

[1] *Grupo de Física Téorica de Materia Condensada, Departamento de Física, Universidad de los Andes, A. A. 4976, Bogotá, Colombia*

[2] *Institute for Theoretical Physics–HAS, Eötvös University, Pázmány Péter sétány 1/a, 1117 Budapest, Hungary*

[3] *Laboratoire Matière et Systèmes Complexes, CNRS UMR 7057, Université Paris Diderot–Paris 7, 10 rue Alice Domon et Léonie Duquet, 75205 Paris cedex 13, France*

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We present a fully microscopics-based calculation of the Casimir effect in a nonequilibrium system, namely, an energy-flux-driven quantum *XX* chain. The force between the walls (transverse-field impurities) is calculated in a nonequilibrium steady state which is prepared by letting the system evolve from an initial state with the two halves of the chain prepared at equilibrium at different temperatures. The steady state emerging in the large-time limit is homogeneous but carries an energy flux. The Casimir force in this nonequilibrium state is calculated analytically in the limit when the transverse fields are small. We find that the the Casimir force range is reduced compared to the equilibrium case, and suggest that the reason for this is the reduction of fluctuations in the flux-carrying steady state.

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I. INTRODUCTION

Historically, the Casimir force [1] is the effective interaction that develops between two ideal conductors in vacuum owing to the quantum fluctuations of the electromagetic field. The zero-temperature case was soon generalized by Lifschitz [2], who also calculated the effective force induced by thermal fluctuations. Since then, as decades elapsed, these theoretical predictions have received qualitative and then quantitative confirmation [3,4]. For the electromagnetic field, the thermal component of the Casimir force is orders of magnitude weaker than its zero-temperature counterpart. This accounts for the relatively late confirmation of the thermal Casimir force [4,5]. The Casimir force has also appeared in other frameworks, such as low-dimensional quantum liquids [6–8]. It was found there, for example, that the Casimir force between magnetic impurities displays Friedel-like oscillations. This large body of existing literature is mainly devoted to equilibrium situations.

In this paper, we wish to investigate the interplay of the long-range effective interactions produced by an energy flux running through the chain and the fluctuation-mediated interactions between two pointlike defects located a distance ℓ apart. Following the accepted terminology, the effective interaction between the defects will be called the Casimir force. We have been inspired by a recent series of works by Antezza *et al.* [10,14] in which a similar problem was considered, namely the calculation of the Casimir force produced by the electromagnetic field in between two parallel plates kept at unequal temperatures. This work followed earlier investigations by Polder and Van Hove [11], recently reviewed by Volokitin and Persson [12] and Dorofeyev [13]. However, in contrast to [10,14], in our study there will be no phenomenological input in the form of linear response coefficients; it will be fully microscopically based, for modeling both the walls and the bulk of the system. Our theoretical laboratory will be an integrable spin chain (the *XX* quantum chain) where similar calculations have been carried out before for purely equilibrium situations [6,7,9]. The results we have obtained differ notably from those obtained in a different framework by Antezza *et al.* [10,14]. In particular, we find that, in contrast to their work, even in the simplest case of two parallel plates, in the presence of an energy flux, the Casimir force does not appear to be the average of the contributions of two equilibrium Casimir forces, corresponding to the two imposed temperatures. We, on the other hand, do not have a simple decomposition owing to the fact that the energy flux is the quantity that governs the value of the force, and in our case it clearly emerges that the presence of the flux decreases the magnitude of the force.

Our results are presented in the following order. Section II briefly reviews the *XX* chain and describes the properties of nonequilibrium steady states arising from preparation of the system with a steplike temperature profile. In Sec. III, we introduce two magnetic impurities and show how the steady-state properties are modified in their presence. In Sec. IV, we derive the Casimir force exerted by one impurity upon the other and present comparisons with earlier phenomenological calculations. Conclusions and future research directions follow in Sec. V. Finally, in Appendix A we present a physicist's derivation of the nonequilibrium steady-state properties, previously obtained via more complex C^* algebraic methods.

II. TRANSVERSE *XX* CHAIN

A. Equilibrium

The transverse *XX* chain is one of the simplest quantum systems displaying long-range correlations and the associated large fluctuations. It is defined by the Hamiltonian

$$\hat{H} = -\sum_i \left(s_i^x s_{i+1}^x + s_i^y s_{i+1}^y + h s_i^z\right), \qquad (1)$$

where $\vec{s}_i = \frac{1}{2}\vec{\sigma}_i$ and σ_i^α ($\alpha = x,y,z$) denote the three Pauli matrices at sites $-N,\ldots,-1,0,\ldots,N \to \infty$ of a $d=1$ chain, h is the transverse field in units of the coupling J, and $J=1$ is set in the following. The standard way [15] to study the transverse *XX* chain is to resort to the Jordan-Wigner transformation, which maps the spin chain onto a one-dimensional system of free fermions with energy spectrum $\varepsilon_q = -\cos q - h$:

$$\hat{H} = \sum_q \varepsilon_q c_q^\dagger c_q, \tag{2}$$

where c_q is the Fourier transform of $c_i = \sigma_i^- [\prod_{j\leqslant i-1}(-\sigma_j^z)]$ with wave numbers in the range $-\pi \leqslant q \leqslant \pi$. As far as large-distance or transport properties are concerned, most of the relevant large-scale physics is governed by the modes q in the vicinity of the Fermi level $\pm\kappa$ determined from $\cos\kappa = -h$. We shall thus often resort to the approximation of effective relativistic fermions, which consists in linearizing the dispersion relation around $\pm\kappa$ (by setting $q = \pm\kappa + k$) and considering the modes with $q > 0$ or $q < 0$ as two independent families of relativistic fermions, namely, the left and the right movers, with velocity $c = \sin\kappa$. Further details of the validity of such a description can be found, for example, in [16]. A phenomenological cutoff Λ is imposed, when needed, on the new k modes.

B. Steady state with energy flux

Our goal is to investigate the nonequilibrium states of $\hat{H}$ that carry a given energy flux $\langle \hat{J}_E\rangle \neq 0$. Here the energy flux $\hat{J}_E$ is given by

$$\hat{J}_E = \sum_q \sin q\, \varepsilon_q c_q^\dagger c_q. \tag{3}$$

One way to achieve a current-carrying steady state has recently been discussed by Ogata [17], who built upon earlier studies by Araki [18], Tasaki [19], and Pillet and Aschbacher [20]. Initially, the chain is prepared in the following way: Its left side, say from $-\infty$ up to site $j = 0$, is in thermal equilibrium at inverse temperature β_1, while its right-hand side from $j = 1$ up to $+\infty$ is in equilibrium at inverse temperature β_2. Both halves are initially disconnected. Then at $t = 0$ contact is made through the $(j = 0, j = 1)$ bond, and the system eventually settles into a nonequilibrium steady state. Because of its infinite thermal conductivity, the temperature profile is flat in the central region, whose extent expands at finite velocity. We shall not be concerned here with the dynamics of the formation of the central region and with the front propagation issues [21–24]. We focus on the asymptotic homogeneous steady state of the expanding central region of the system, where the fermion occupation number in the steady state is given by

$$\langle c_k^\dagger c_k\rangle = F_k = \frac{\theta(k)}{1+e^{\beta_1\varepsilon_k}} + \frac{\theta(-k)}{1+e^{\beta_2\varepsilon_k}}. \tag{4}$$

The above result can be derived mathematically rigorously [17]; a physicist's proof of (4) is provided in Appendix A. It will often prove convenient to introduce $\beta = \frac{\beta_1+\beta_2}{2}$ and $\varepsilon_k' = \theta(k)\frac{\beta_1}{\beta}\varepsilon_k + \theta(-k)\frac{\beta_2}{\beta}\varepsilon_k$, so that F_k appears as the effective Fermi-Dirac occupation number at temperature β of free fermions with energy spectrum ε_k'. The discontinuity of ε_k' is the consequence of the long-range effective interactions in the steady state, as discussed in [17]. Ogata [17] has also discussed the dependence of $\langle \hat{J}_E\rangle$ on h, β_1, and β_2 in light of the experimental literature [27]. We note that there exist other ways to produce a nonequilibrium steady state (see [21,25,26]), but we shall not consider these here [28].

III. A SPIN CHAIN WITH TWO IMPURITIES

A. About the force

In our case, the walls of the Casimir setup will be two magnetic impurities at lattice sites $\mp\ell/2$ with strength $\delta h_{1/2}$ which enter the Hamiltonian through additional terms of the form

$$\delta\hat{H} = -\delta h_1 c_{-\ell/2}^\dagger c_{-\ell/2} - \delta h_2 c_{\ell/2}^\dagger c_{\ell/2}. \tag{5}$$

The task is to determine the effective force between those impurities. Since we are ultimately interested in a nonequilibrium setting, we must circumvent the methods used before in [6,7,29–31] that relied on determining a ground-state energy or a free energy. We adopt the following definition of the force on a lattice. Let $\hat{H}_j$ be the energy density at site j,

$$\hat{H}_j = -\tfrac{1}{2}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - h c_j^\dagger c_j. \tag{6}$$

The force F experienced by the defect located at site $-\ell/2$ is given by

$$F = \tfrac{1}{2}\langle \hat{H}_{-\ell/2-1} - \hat{H}_{-\ell/2+1}\rangle, \tag{7}$$

where the average is over the nonequilibrium steady state. Since the local energy is quadratic in the fermionic operators, it is clear that, in order to calculate F, it is sufficient to know the two-point equal-time Green's function $\mathcal{G}_{ij}(t,t') = \langle c_i(t)c_j^\dagger(t)\rangle$ as calculated in the next section. Note that there is some arbitrariness in defining a force directly on the lattice. In a continuum limit, for example, when one focuses on wave vectors close to the Fermi level, such differences become irrelevant, and our definition matches the expression of the force one would obtain directly from a continuum theory.

B. Green's functions

In order to determine the Green's function $\mathcal{G}_{ij}(t,t') = \langle c_i(t)c_j^\dagger(t')\rangle$, we shall need another Green's function $G_{ij}(t,t')$ introduced through the Heisenberg picture,

$$c_i(t) = \sum_n G_{in}(t,0)c_n(0), \quad c_j^\dagger(t) = \sum_m G_{mj}^*(t,0)c_m^\dagger(0), \tag{8}$$

which leads to

$$\mathcal{G}_{ij}(t,t') = \sum_{m,n} G_{in}(t,0)G_{mj}^*(t',0)\langle c_n(0)c_m^\dagger(0)\rangle. \tag{9}$$

In the latter formula, we take as the initial state the stationary state. The Green's function G has to be calculated in the presence of defects, which can be carried out by a variety of methods, one of them being presented in Appendix B. The results can most conveniently be expressed in terms of the Green's function $g_{mn}(\omega)$ in the absence of defects,

$$g_{mn}(\omega) = \int \frac{dq}{2\pi}\frac{e^{-iq(m-n)}}{-i\omega+\varepsilon_q}, \tag{10}$$

where ω is conjugate to it. We find that the Fourier transform of G_{ij} reads, in the weak-magnetic-defect limit,

$$\begin{aligned} G_{ij} &= g_{ij} + \delta h_1 g_{i-}g_{-j} + \delta h_2 g_{i+}g_{+j} \\ &\quad + \delta h_1^2 g_{i-}g_{-j}g_{--} + \delta h_2^2 g_{i+}g_{+j}g_{++} \\ &\quad + \delta h_1\delta h_2(g_{i+}g_{-j}g_{+-} + g_{i-}g_{+j}g_{-+}) + O(\delta h^3), \end{aligned} \tag{11}$$

where $\pm$ is for $\pm\frac{\ell}{2}$. The alternative limit $\delta h_1,\delta h_2 \to \infty$ would correspond to freezing the degrees of freedom of the defects, which would bring us closer to the original setting of Casimir. In terms of space Fourier transforms, this expression becomes

$$\begin{aligned} G(q,q',\omega) &= g_q\delta_{q,q'} + (\delta h_1 e^{i(q'-q)\ell/2} + \delta h_2 e^{-i(q'-q)\ell/2}) g_q g_{q'} \\ &\quad + \int \frac{dk}{2\pi}(\delta h_1 e^{i(q'-k)\ell/2} + \delta h_2 e^{-i(q'-k)\ell/2}) \\ &\quad \times (\delta h_1 e^{i(k-q)\ell/2} + \delta h_2 e^{-i(k-q)\ell/2}) g_q g_{q'} g_k \\ &\quad + O(\delta h^3). \end{aligned} \quad (12)$$

We now use that the steady-state Green's function $\mathcal{G}$ is given in (9), where the initial state $\langle c_n(0)c_m^\dagger(0)\rangle$ is the current-carrying steady state obtained from a step temperature profile at $t=-\infty$ and G includes the presence of weak impurities as given in (11). In the infinite time limit, we show in Appendix B that

$$\begin{aligned} &\mathcal{G}(q,q',t) \\ &\simeq F_Q\delta_{q,q'} - (\delta h_1 e^{i(q-q')\ell/2} + \delta h_2 e^{-i(q-q')\ell/2}) \frac{F_q - F_{q'}}{\varepsilon_q - \varepsilon'_q} \\ &\quad + \int \frac{dk}{2\pi}(\delta h_1 e^{i(q'-k)\ell/2} + \delta h_2 e^{-i(q'-k)\ell/2}) \\ &\quad \times (\delta h_1 e^{i(k-q)\ell/2} + \delta h_2 e^{-i(k-q)\ell/2}) \\ &\quad \times \frac{1}{\varepsilon_q - \varepsilon_{q'}}\left(\frac{F_q - F_k}{\varepsilon_q - \varepsilon_k} - \frac{F_{q'} - F_k}{\varepsilon_{q'} - \varepsilon_k}\right). \end{aligned} \quad (13)$$

Since we are interested only in the interactions between the defects, only the cross $\delta h_1 \times \delta h_2$ term contributes to $\mathcal{G}_\times(q,q')$ when the corresponding force is calculated:

$$\begin{aligned} \mathcal{G}_\times(q,q') &= 2\delta h_1 \delta h_2 \int \frac{dk}{2\pi} \frac{\cos(q+q'-2k)\ell/2}{\varepsilon_q - \varepsilon_{q'}} \\ &\quad \times \left(\frac{F_q - F_k}{\varepsilon_q - \varepsilon_k} - \frac{F_{q'} - F_k}{\varepsilon_{q'} - \varepsilon_k}\right). \end{aligned} \quad (14)$$

It is useful to introduce the function $\gamma(r,\omega) = \int \frac{dq}{2\pi} e^{iqr}\gamma_q(\omega)$ whose Fourier transform is defined by

$$\gamma_q(\omega) = \frac{1}{-i\omega + \varepsilon'_q}. \quad (15)$$

With this notation, we have

$$\begin{aligned} \mathcal{G}_\times(q,q') &= 2\delta h_1 \delta h_2 \int \frac{d\omega}{2\pi}\frac{dk}{2\pi} \\ &\quad \times \cos[(q+q'-2k)\ell/2]\gamma_q(\omega)\gamma_{q'}(\omega)\gamma_k(\omega). \end{aligned} \quad (16)$$

In real space, this becomes

$$\begin{aligned} \mathcal{G}_\times(x,y) = \delta h_1 \delta h_2 \int \frac{d\omega}{2\pi} &[\gamma(\ell/2 - x,\omega)\gamma(\ell/2 + y,\omega)\gamma(-\ell,\omega) \\ &+ \gamma(-\ell/2 - x,\omega)\gamma(-\ell/2 + y,\omega)\gamma(\ell,\omega)], \end{aligned} \quad (17)$$

and this form will be used below in the calculation of the force.

IV. CASIMIR EFFECT: THE EFFECTIVE FORCE BETWEEN THE IMPURITIES

A. Formal expression of the force

In this section, we derive the expression of the force in the limit where the impurities are a large distance ℓ apart. In this limit, many of the expressions encountered above simplify significantly. Most notably, the function $\gamma(r,\omega)$, which enters the final expression of the Green's function, reads

$$\gamma(r,\omega) = e^{i\kappa r}\gamma_1(r,\omega) + e^{-i\kappa r}\gamma_2(r,\omega), \quad (18)$$

where we used the following definitions:

$$\gamma_1(r,\omega) = i\frac{e^{-|r\omega|/c_1}}{c_1}[\theta(r)\theta(\omega) - \theta(-r)\theta(-\omega)] \quad (19)$$

and

$$\gamma_2(r,\omega) = i\frac{e^{-|r\omega|/c_2}}{c_2}[-\theta(r)\theta(-\omega) + \theta(-r)\theta(\omega)] \quad (20)$$

with $c_1 = \frac{\beta_1}{\beta}\sin\kappa$ and $c_2 = \frac{\beta_2}{\beta}\sin\kappa$. In Eq. (18), the right- (1) and left- (2) moving fermions account for the c_1- and c_2-dependent terms, respectively. In terms of $\gamma_1(r,\omega)$ and $\gamma_2(r,\omega)$, the force can be written as

$$\begin{aligned} F &= \frac{i\,c_1}{2\,\beta}\left[\sum_\omega(-\partial_y + \partial_x)e^{-i\kappa(x-y)}\Gamma_1(x,y,\omega)\right]_{x=y=-(\ell/2)^-}^{x=y=-(\ell/2)^+} \\ &\quad - \frac{i\,c_2}{2\,\beta}\left[\sum_\omega(-\partial_y + \partial_x)e^{i\kappa(x-y)}\Gamma_2(x,y,\omega)\right]_{x=y=-(\ell/2)^-}^{x=y=-(\ell/2)^+}, \end{aligned} \quad (21)$$

where the frequency sum is over the $\omega = \frac{2n+1}{\beta}\pi$, and where

$$\begin{aligned} \Gamma_1(x,y,\omega) &= \delta h_1 \delta h_2[e^{-2i\kappa\ell}\gamma_2(\ell,\omega)\gamma_1(y - \ell/2,\omega) \\ &\quad \times \gamma_1(-x - \ell/2,\omega) + e^{2i\kappa\ell}\gamma_2(-\ell,\omega) \\ &\quad \times \gamma_1(y + \ell/2,\omega)\gamma_1(-x + \ell/2,\omega)] \end{aligned} \quad (22)$$

and

$$\begin{aligned} \Gamma_2(x,y,\omega) &= \delta h_1 \delta h_2[e^{2i\kappa\ell}\gamma_1(\ell,\omega)\gamma_2(y - \ell/2,\omega) \\ &\quad \times \gamma_2(-x - \ell/2,\omega) + e^{-2i\kappa\ell}\gamma_1(-\ell,\omega) \\ &\quad \times \gamma_2(y + \ell/2,\omega)\gamma_2(-x + \ell/2,\omega)]. \end{aligned} \quad (23)$$

Finally, in the limit $\ell \gg 1$, the force between impurities given by Eq. (7) is the discontinuity of the energy density across the defect, and in terms of $\gamma_1(r,\omega)$ and $\gamma_2(r,\omega)$, it can be written as

$$\begin{aligned} F &= \frac{4\delta h_1 \delta h_2 \kappa \sin(2\kappa\,|\ell|)}{\beta}\sum_\omega \gamma_1(\ell,\omega)\gamma_2(-\ell,\omega) \\ &\quad + \frac{2\delta h_1 \delta h_2 \cos(2\kappa\ell)}{\beta}\sum_\omega \omega\left(\frac{1}{c_1} + \frac{1}{c_2}\right)\gamma_1(\ell,\omega)\gamma_2(-\ell,\omega). \end{aligned} \quad (24)$$

Note that the force can be written in terms of the unperturbed Green's functions of the left- and right-moving fermions. The expression of the force (24) is the central result of this work. We now specify this result, first to a known situation to make contact with existing results, and second to the physically more interesting case of a current-carrying chain.

B. Equilibrium (recovering existing results)

In [7] the authors studied the Casimir forces between defects in equilibrium one-dimensional quantum liquids at zero temperature; in order to reproduce their results we have to take $\beta_1 = \beta_2 = \beta$ with $\beta \to \infty$ in Eq. (24). In this limit, the sum over ω can be changed by an integral $\frac{1}{\beta}\sum_\omega \to \int \frac{d\omega}{2\pi}$, and after some algebra we find

$$F = -\delta h_1 \delta h_2 \left(\frac{\kappa \sin(2\kappa\ell)}{\pi \ell \sin\kappa} + \frac{\cos(2\kappa\ell)}{2\pi \ell^2 \sin\kappa} \right), \tag{25}$$

which is indeed the interaction force associated with the interaction potential between impurities given by Eq. (18) of Ref. [7]. We conclude that in the case of zero temperature the leading term of the interaction force decays as $1/\ell$ and it oscillates with wavelength π/κ.

C. Out of equilibrium, with a heat flux

In the case $\beta_1 \neq \beta_2$, a heat flux drives the spin chain into a nonequilibrium steady state. The expression for the force reads

$$F = -\frac{4\delta h_1 \delta h_2 \kappa \sin(2\kappa\,|\ell|)}{\beta c_1 c_2} \left(\frac{1}{e^{|\ell|\pi p/\beta} - e^{-|\ell|\pi p/\beta}} \right) - \frac{4\pi \delta h_1 \delta h_2 p \cos(2\kappa\ell)}{\beta^2 c_1 c_2} \frac{e^{\frac{|\ell|\pi p}{\beta}}}{(e^{|\ell|\pi p/\beta} - e^{-|\ell|\pi p/\beta})^2}, \tag{26}$$

where $p = \frac{1}{c_1} + \frac{1}{c_2}$ and we recall that $c_1 = \frac{\beta_1}{\beta}\sin\kappa$ and $c_2 = \frac{\beta_2}{\beta}\sin\kappa$. As one can see, the force decays exponentially with a characteristic length $\xi = \frac{\beta}{\pi p}$. This is in contrast with the zero-temperature case, where the decay is algebraic. The oscillatory factors $\sin(2\kappa\,|\ell|)$ and $\cos(2\kappa\ell)$ are the same as in the zero-temperature limit. As expected, the $\beta_1 = \beta_2 = \beta$ thermal equilibrium limit is consistent with previously found results [7].

V. CONCLUSIONS

We have fully determined the Casimir force between two magnetic-field defects of weak amplitude in the nonequilibrium steady state of the *XX* spin chain carrying an energy flux. The overall qualitative behavior is similar to that obtained from equilibrium thermal fluctuations: The Casimir force decays exponentially with increasing distance between the impurities. However, that decay is notably sharper in the presence of an energy flux than without. Thus we conclude that the presence of the energy flux tends to weaken the Casimir force. This can be seen from the correlation length $\xi = \frac{\beta}{\pi p}$ since, as the system approaches equilibrium $\beta_1 \to \beta_2$, $\xi_{\text{out of eq}} - \xi_{\text{eq}} = -\frac{c(\beta_1-\beta_2)^2}{8\pi\beta}$, and thus $\xi_{\text{out of eq}} < \xi_{\text{eq}}$. This strengthens the general picture [30,31] that nonequilibrium fluctuations lead to stiffer systems, which, as the present calculation reveals, do not favor fluctuation-mediated interactions.

It is expected that the leading behavior is different in the presence of strong impurities. And so it would be interesting to push our investigations further to probe the similarities and the differences with the calculation of Antezza *et al.* [10] bearing on the electromagnetic field between two plates thermalized at unequal temperatures. The conceptual issue here is whether different "ensembles" with either a fixed temperature difference or a fixed energy flux should lead to the same physical results in a nonequilibrium setting. Another issue of interest is related to our spin chain having an infinite conductivity. The question of what happens in realistic systems with nonintegrable interactions is an open one that we would like to address in the future.

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APPENDIX A: HEAT BATHS AT DIFFERENT TEMPERATURES

In her work [17], Ogata resorts to algebraic methods (C^* algebras), to derive the steady-state properties of the spin chain whose two semi-infinite halves are initially prepared in thermalized states at T_1 and T_2. The goal of this appendix is to rederive her results sticking to the standard methods familiar to physicists.

We start from two semi-infinite XX spin chains defined for $n \leqslant 0$ and $n \geqslant 1$, which are respectively thermalized at inverse temperatures β_1 and β_2. The Hamiltonian for the left-hand side reads

$$\hat{H}^{(1)} = \sum_{n\leqslant -1} \left(-\frac{1}{2} c_n^\dagger c_{n+1} - \frac{1}{2} c_{n+1}^\dagger c_n - h c_n^\dagger c_n \right) = \frac{2}{\pi}\int_0^\pi dq\, \varepsilon_q c_q^\dagger c_q, \tag{A1}$$

where the Fourier transform is given by $c_q = -i\sqrt{\frac{2}{\pi}}\sum_{n\leqslant 0} \sin[(n-1)q]c_n$ $(0 \leqslant q \leqslant \pi)$ and the energy spectrum is $\varepsilon_q = -\cos q - h$. For the right-hand side, we also have

$$\hat{H}^{(2)} = \sum_{n\geqslant 0} \left(-\frac{1}{2} c_n^\dagger c_{n+1} - \frac{1}{2} c_{n+1}^\dagger c_n - h c_n^\dagger c_n \right) = \frac{2}{\pi}\int_0^\pi dq\, \varepsilon_q c_q^\dagger c_q, \tag{A2}$$

where the Fourier transform is now given by $c_q = -i\sqrt{\frac{2}{\pi}}\sum_{n\geqslant 1} \sin(nq)c_n$ $(0 \leqslant q \leqslant \pi)$ and the energy spectrum is of course the same. In the initial state of the left-hand side of the chain, we have that

$$\langle c_n^\dagger c_m \rangle = \frac{2}{\pi}\int_0^\pi dq \sin[q(n-1)] \sin[q(m-1)] f_q^{(1)}, \quad n,m \leqslant 0, \tag{A3}$$

where $f_q^{(1)} = \frac{1}{e^{\beta_1 \varepsilon_q}+1}$ is the Fermi-Dirac occupation number. Similarly, for the right-hand side, we have

$$\langle c_n^\dagger c_m \rangle = \frac{2}{\pi}\int_0^\pi dq \sin(qn)\sin(qm) f_q^{(2)}, \quad n,m \geqslant 1, \tag{A4}$$

with $f_q^{(2)} = \frac{1}{e^{\beta_2 \varepsilon_q}+1}$. Ogata's result [17] states that, in the steady state reached in the large-time limit,

$$\langle c_n^\dagger c_m \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(m-n)} F_k, \tag{A5}$$

where the occupation number F_k is given by

$$F_k = \frac{1}{e^{\beta_1 \varepsilon_k}+1}\theta(k) + \frac{1}{e^{\beta_2 \varepsilon_k}+1}\theta(-k). \tag{A6}$$

In order to prove the result (A6), we start by determining the time-dependent Green's function for arbitrary lattice sites. Since $c_k(t) = e^{i\varepsilon_k t} c_k(0)$ (the Fourier modes $-\pi \leqslant k \leqslant \pi$ refer to the whole translationally invariant chain), we have that

$$c_n(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikn} c_k(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikn+i\varepsilon_k t} \sum_j e^{+ikj} c_j(0), \tag{A7}$$

with a similar relationship for $c_m^\dagger(t)$; hence the Green's function evaluated at equal—but finite—times reads

$$\langle c_m^\dagger(t) c_n(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{dk'}{2\pi} e^{ik'm-ikn-i\varepsilon_{k'}t+i\varepsilon_k t} \times \sum_{j,\ell} e^{-ik'\ell+ikj} \langle c_\ell^\dagger(0) c_j(0)\rangle. \tag{A8}$$

The angular brackets in the average appearing in the right-hand side denote the sampling with respect to the thermalized initial state. We know that in this initial state

$$\langle c_\ell^\dagger(0) c_j(0)\rangle = \begin{cases} \frac{2}{\pi}\int_0^\pi dq \, \sin(\ell q)\sin(jq) f_q^{(2)} & \text{if } \ell, j \geqslant 1, \\ \frac{2}{\pi}\int_0^\pi dq \, \sin[(\ell-1)q]\sin[(j-1)q] f_q^{(1)} & \text{if } \ell, j \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{A9}$$

It is convenient to rewrite

$$\langle c_m^\dagger(t) c_n(t) \rangle = \langle c_m^\dagger(t) c_n(t) \rangle_{(1)} + \langle c_m^\dagger(t) c_n(t) \rangle_{(2)}, \tag{A10}$$

where

$$\langle c_m^\dagger(t) c_n(t) \rangle_{(1)} = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{dk'}{2\pi} e^{ik'm-ikn-i\varepsilon_{k'}t+i\varepsilon_k t} \times \sum_{j,\ell \leqslant 0} e^{-ik'\ell+ikj} \langle c_\ell^\dagger(0) c_j(0)\rangle,$$

$$\langle c_m^\dagger(t) c_n(t) \rangle_{(2)} = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{dk'}{2\pi} e^{ik'm-ikn-i\varepsilon_{k'}t+i\varepsilon_k t} \times \sum_{j,\ell \geqslant 1} e^{-ik'\ell+ikj} \langle c_\ell^\dagger(0) c_j(0)\rangle. \tag{A11}$$

Let us focus, say, on $\langle c_m^\dagger(t) c_n(t) \rangle_{(2)}$:

$$\begin{aligned}\langle c_m^\dagger(t) c_n(t) \rangle_{(2)} &= \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{dk'}{2\pi} e^{ik'm-ikn-i\varepsilon_{k'}t+i\varepsilon_k t} \times \sum_{j,\ell \geqslant 1} e^{-ik'\ell+ikj} \langle c_\ell^\dagger(0) c_j(0)\rangle \\ &= \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{dk'}{2\pi} e^{ik'm-ikn-i\varepsilon_{k'}t+i\varepsilon_k t} \times \sum_{j,\ell \geqslant 1} e^{-ik'\ell+ikj} \frac{2}{\pi}\int_0^\pi dq \, \sin(\ell q)\sin(jq) f_q^{(2)}.\end{aligned} \tag{A12}$$

Now, we make use of the following identities, valid as $\delta \to 0^+$, in terms of distributions:

$$\sum_{j \geqslant 1} \sin qj e^{\pm ikj} = \lim_{\delta \to 0^+} \frac{1}{2} \frac{\sin q}{\cos k - \cos q \mp i\delta \mathrm{sgn}(k)}; \tag{A13}$$

hence we get, the limit $\delta \to 0^+$ being understood, that

$$\langle c_m^\dagger(t) c_n(t) \rangle_{(2)} = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{dk'}{2\pi} e^{ik'm-ikn-i\varepsilon_{k'}t+i\varepsilon_k t} F(k,k'), \tag{A14}$$

where we have introduced the notation

$$F(k,k') = \frac{2}{\pi}\int_0^\pi dq \, f_q^{(2)} \frac{1}{2}\frac{\sin q}{\cos k - \cos q - i\delta \mathrm{sgn}(k)} \times \frac{1}{2}\frac{\sin q}{\cos k' - \cos q + i\delta \mathrm{sgn}(k')}. \tag{A15}$$

We now focus on the q integral that appears in $F(k,k')$:

$$\begin{aligned} F(k,k') &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{\sin^2 q f_q^{(2)}}{[\cos k - \cos q - i\delta \mathrm{sgn}(k)][\cos k' - \cos q + i\delta \mathrm{sgn}(k')]} \\ &= \frac{1}{\cos k - \cos k' - i\delta[\mathrm{sgn}(k) + \mathrm{sgn}(k')]} \oint \frac{dz}{2\pi i} \sin^2 q f_q^{(2)} \\ &\quad \times \left(\frac{1}{1+z^2-2z[\cos k - i\delta \mathrm{sgn}(k)]} - \frac{1}{1+z^2-2z[\cos k' + i\delta \mathrm{sgn}(k')]} \right), \end{aligned} \tag{A16}$$

where we have set $z = e^{iq}$ and the z integral runs counterclockwise around the unit circle. Explicitly carrying out the z integral, we obtain

$$F(k,k') = \frac{i}{2}\frac{1}{\cos k - \cos k' - i\delta[\mathrm{sgn}(k) + \mathrm{sgn}(k')]} \times \left(f_{k'}^{(2)} \sin k' + f_k^{(2)} \sin k \right). \tag{A17}$$

We are thus left with evaluating

$$\langle c_m^\dagger(t)c_n(t)\rangle_{(2)} = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\frac{dk'}{2\pi}e^{ik'm-ikn-i\varepsilon_{k'}t+i\varepsilon_k t} \times \frac{i}{2}\frac{1}{\cos k-\cos k'-i\delta[\text{sgn}(k)+\text{sgn}(k')]} \times\left(f_{k'}^{(2)}\sin k'+f_k^{(2)}\sin k\right). \tag{A18}$$

In order to extract the long-time behavior of (A18), we change variables from k to $u=(k-k')t\sin k'$ and expand in powers of $1/t$, keeping only the leading order at fixed m and n:

$$\lim_{t\to\infty}\langle c_m^\dagger(t)c_n(t)\rangle_{(2)} = \int_{-\pi}^{\pi}\frac{dk'}{2\pi}e^{ik'(m-n)}f_{k'}^{(2)}i \times\int_{-\infty}^{+\infty}\frac{du}{2\pi}\frac{e^{iu}}{-u-i\,\text{sgn}(k')} = \int_{-\pi}^{\pi}\frac{dk'}{2\pi}e^{ik'(m-n)}f_{k'}^{(2)}\theta(-k'). \tag{A19}$$

A similar reasoning leads to

$$\lim_{t\to\infty}\langle c_m^\dagger(t)c_n(t)\rangle_{(1)} = \int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{ik(m-n)}f_k^{(1)}\theta(k). \tag{A20}$$

Hence, (A19) together with (A20) are exactly the announced result (A5) and (A6).

APPENDIX B: GREEN'S FUNCTIONS

1. Green's functions for the evolution

In order to determine $\mathcal{G}_{ij}(t,t')$, we introduce the Green's function $G_{ij}(t,t')$ using the Heisenberg picture,

$$c_i(t)=\sum_n G_{in}(t,0)c_n(0),\quad c_j^\dagger(t)=\sum_m G_{mj}^*(t,0)c_m^\dagger(0), \tag{B1}$$

which leads to $\mathcal{G}_{ij}(t,t')=\sum_{m,n}G_{in}(t,0)G_{mj}^*(t',0)\langle c_n(0)c_m^\dagger(0)\rangle$. In the latter formula, we take as the initial state the stationary state. The Green's function G in the presence of defects can be determined by a variety of methods. We choose to use a path-integral formulation. Going to Grassmann fields, the Fourier transform of $\mathcal{G}_{ij}(t,t')$ is given by

$$G_{ij}(\omega)=\langle c_i(\omega)\bar{c}_j(\omega)\rangle = \frac{\int\mathcal{D}\bar{c}\mathcal{D}c\,c_i(\omega)\bar{c}_j(\omega)e^{-S[\bar{c},c]}}{\int\mathcal{D}\bar{c}\mathcal{D}c e^{-S[\bar{c},c]}}, \tag{B2}$$

where the action $S=S_0+\delta S$ is as follows:

$$S_0[\bar{c},c]=\sum_\omega\sum_i\left[(-i\omega-h)\bar{c}_i(\omega)c_i(\omega) -\frac{1}{2}\bar{c}_ic_{i+1}-\frac{1}{2}\bar{c}_ic_{i-1}\right] \tag{B3}$$

and

$$\delta S[\bar{c},c]=-\sum_{i,\omega}\delta h_i\bar{c}_ic_i \tag{B4}$$

and the magnetic field is $\delta h_i=\delta h_+\delta_{i,-\ell/2}+\delta h_-\delta_{i,\ell/2}$. We introduce $g_{ij}(\omega)$, the Green's function in the absence of defects as given by S_0. We start with the definition of the Green's function:

$$\langle c_i(\omega)\bar{c}_j(\omega)\rangle = \frac{\int\mathcal{D}\bar{c}\mathcal{D}c\,c_i(\omega)\bar{c}_j(\omega)e^{-S[\bar{c},c]}}{\int\mathcal{D}\bar{c}\mathcal{D}c e^{-S[\bar{c},c]}}, = \frac{1}{Z[0,0]}\frac{\delta}{\delta\eta_j}\frac{\delta}{\delta\bar{\eta}_i}Z[\bar{\eta},\eta]\Big|_{\bar{\eta}=\eta=0} \tag{B5}$$

where η and $\bar{\eta}$ are introduced through the partition function

$$Z[\bar{\eta},\eta] =\int\mathcal{D}\bar{c}\mathcal{D}c\exp\left(-S_0[\bar{c},c]+\sum_{j,\omega}(\delta h_j\bar{c}_jc_j+\bar{\eta}_jc_j+\bar{c}_j\eta_j)\right). \tag{B6}$$

Then we use the representation

$$\exp\left(\sum_{j,\omega}\delta h_j\bar{c}_jc_j\right) =\int\prod_j\mathcal{D}\bar{\phi}_j\mathcal{D}\phi_j \times\exp\left(-\sum_{j,\omega}[\bar{\phi}_j\phi_j+\sqrt{\delta h_j}(\bar{\phi}_jc_j+\bar{c}_j\phi_j)]\right), \tag{B7}$$

giving the partition function as

$$\begin{aligned}Z[\bar{\eta},\eta] &= \int\mathcal{D}\bar{\phi}\mathcal{D}\phi\exp\left(-\sum_{j,\omega}\bar{\phi}_j\phi_j\right) \times\int\mathcal{D}\bar{c}\mathcal{D}c\exp\Big(-S_0[\bar{c},c]+\sum_{j,\omega}[\bar{c}_j(\eta_j +\sqrt{\delta h_j}\phi_j)+(\bar{\eta}_j+\sqrt{\delta h_j}\bar{\phi}_j)c_j]\Big)\\ &=\int\mathcal{D}\bar{\phi}\mathcal{D}\phi\exp\left(-\sum_{j,\omega}\bar{\phi}_j\phi_j\right)(\det g_{ij}) \times\exp\left(\sum_{i,j,\omega}(\bar{\eta}_i+\sqrt{\delta h_i}\bar{\phi}_i)g_{ij}(\eta_j+\sqrt{\delta h_j}\phi_j)\right)\\ &=\det g_{ij}e^{\bar{\eta}_ig_{ij}\eta_j}\int\mathcal{D}\bar{\phi}\mathcal{D}\phi \times\exp\left(-\bar{\phi}_i\gamma_{ij}\phi_j+\sum_{j,\omega}(\bar{\xi}_j\phi_j+\bar{\phi}_j\xi_j)\right),\end{aligned} \tag{B8}$$

where we have defined $\gamma_{ij}=\delta_{ij}-\sqrt{\delta h_i\delta h_j}g_{ij}$ and $\xi_m=\sqrt{\delta h_m}\sum_j g_{mj}\eta_j$, $\bar{\xi}_n=\sqrt{\delta h_n}\sum_i g_{in}\bar{\eta}_i$. We perform the remaining path integral and obtain

$$Z[\bar{\eta},\eta]=\det g_{ij}\det\gamma_{ij}\exp\left(\bar{\eta}_ig_{ij}\eta_j+\sum_{n,m,\omega}\bar{\xi}_n[\gamma^{-1}]_{nm}\xi_m\right), \tag{B9}$$

where we have defined $\gamma_{ij} = \delta_{ij} - \sqrt{\delta h_i \delta h_j} g_{ij}$ and $\xi_m = \sqrt{\delta h_m} \sum_j g_{mj} \eta_j$, $\bar{\xi}_n = \sqrt{\delta h_n} \sum_i g_{in} \bar{\eta}_i$. The Green's function can be now found from the coefficient of $\bar{\eta}_i \eta_j$ in the exponential that appears in (B9), so that

$$G_{ij} = g_{ij} + \sum_{m,n} \sqrt{\delta h_n \delta h_m} g_{in} g_{mj} [\gamma^{-1}]_{nm} = g_{ij} + \frac{\delta h_+ (1 - \delta h_- g_{--}) g_{i+} g_{+j} + \delta h_- (1 - \delta h_+ g_{++}) g_{i-} g_{-j}}{(1 - \delta h_+ g_{++})(1 - \delta h_- g_{--}) - \delta h_+ \delta h_- g_{-+} g_{+-}} + \frac{\delta h_+ \delta h_- (g_{i+} g_{-j} g_{+-} + g_{i-} g_{+j} g_{-+})}{(1 - \delta h_+ g_{++})(1 - \delta h_- g_{--}) - \delta h_+ \delta h_- g_{-+} g_{+-}}, \tag{B10}$$

where we used the shorthand notation $\pm$ for $\pm \frac{\ell}{2}$. In the weak-magnetic-defect limit of interest here, we have

$$G_{ij} = g_{ij} + \delta h_1 g_{i-} g_{-j} + \delta h_2 g_{i+} g_{+j} + \delta h_1^2 g_{i-} g_{-j} g_{--} + \delta h_2^2 g_{i+} g_{+j} g_{++} + \delta h_1 \delta h_2 (g_{i+} g_{-j} g_{+-} + g_{i-} g_{+j} g_{-+}) + \cdots. \tag{B11}$$

It will prove more convenient to resort to the Fourier transforms of these Green's functions,

$$G(q,q',\omega) = g_q \delta_{q,q'} + (\delta h_1 e^{i(q'-q)\ell/2} + \delta h_2 e^{-i(q'-q)\ell/2}) g_q g_{q'} + \int \frac{dk}{2\pi} (\delta h_1 e^{i(q'-k)\ell/2} + \delta h_2 e^{-i(q'-k)\ell/2}) \times (\delta h_1 e^{i(k-q)\ell/2} + \delta h_2 e^{-i(k-q)\ell/2}) g_q g_{q'} g_k + \cdots. \tag{B12}$$

2. Steady-state Green's function

In order to calculate the steady-state Green's function, we start with the expression for $\mathcal{G}_{ij}(t,t')$ given right after (B1) in which the initial state is the current-carrying steady state obtained from a step temperature profile at $t = -\infty$. After Ogata [17], if one initially prepares the spin chain with its left-hand side at inverse temperature β_1 and right-hand side at inverse temperature β_2, one has that in the steady state

$$\langle c_n^\dagger(0) c_m(0) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(m-n)} F_k \tag{B13}$$

with the occupation number F_k given by (4). We now set out to determine $\mathcal{G}$ explicitly, in the weak-impurity-strength regime. In Fourier space we have that

$$\mathcal{G}(Q,Q',t) = \int \frac{dq}{2\pi} G(Q,q,t) G^*(Q',q,t) F_q. \tag{B14}$$

In the weak-impurity limit, given that

$$G(q,q',t) = e^{i\varepsilon_q t} \delta_{q,q'} - (\delta h_1 e^{i(q-q')\ell/2} + \delta h_2 e^{-i(q-q')\ell/2}) \frac{e^{i\varepsilon_q t} - e^{i\varepsilon_{q'} t}}{\varepsilon_q - \varepsilon_q'} + \int \frac{dk}{2\pi} (\delta h_1 e^{i(q'-k)\ell/2} + \delta h_2 e^{-i(q'-k)\ell/2}) \times (\delta h_1 e^{i(k-q)\ell/2} + \delta h_2 e^{-i(k-q)\ell/2}) \times \frac{1}{\varepsilon_q - \varepsilon_{q'}} \left(\frac{e^{i\varepsilon_q t} - e^{i\varepsilon_k t}}{\varepsilon_q - \varepsilon_k} - \frac{e^{i\varepsilon_{q'} t} - e^{i\varepsilon_k t}}{\varepsilon_{q'} - \varepsilon_k} \right) \tag{B15}$$

and retaining only the leading (time-independent) behavior in the large-time limit, we arrive at

$$\mathcal{G}(Q,Q',t) \simeq F_Q \delta_{Q,Q'} - (\delta h_1 e^{i(Q-Q')\ell/2} + \delta h_2 e^{-i(Q-Q')\ell/2}) \frac{F_Q - F_{Q'}}{\varepsilon_Q - \varepsilon_Q'} + \int \frac{dk}{2\pi} (\delta h_1 e^{i(Q'-k)\ell/2} + \delta h_2 e^{-i(Q'-k)\ell/2}) \times (\delta h_1 e^{i(k-Q)\ell/2} + \delta h_2 e^{-i(k-Q)\ell/2}) \times \frac{1}{\varepsilon_Q - \varepsilon_{Q'}} \left(\frac{F_Q - F_k}{\varepsilon_Q - \varepsilon_k} - \frac{F_{Q'} - F_k}{\varepsilon_{Q'} - \varepsilon_k} \right). \tag{B16}$$

The above formula is at the basis of our calculation of the force.

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