Order Statistics For i.i.d. Variables

(Dated: November 24, 2013)

I. INTRODUCTION WITH REFERENCES TO EVS IN GENERAL

Extreme value statistics (EVS) was first developed in mathematics [1–3]. Its importance was soon recognized and emphasized in engineering [4, 5], followed by finance and environmental problems [6–10]. Although applications in physics appeared relatively late, they cover a wide range of fields including cosmology [11, 12], spin glasses [13], random fragmentation [14], percolation [15], random matrices [16], and, most actively studied at present, interface fluctuations [17–24].

The extreme value in a batch of data is important, but its study makes use of only a small fraction of the available information. Accordingly, there have been various attempts to extend studies towards near extreme characteristics, such as density of states near extremes [25, 26], first-passage and return-time statistics [28, 29], persistence [30], and record statistics [31–33]. A natural extension (which will be the concern in this lecture) is to consider not only the extreme, but the sequence $x_1, x_2, ..., x_k, ...$ of the 1^{st}, 2^{nd}, ..., k^{th},... largest, i.e. extract information from the order statistics of the system.

Order statistics has been much studied in mathematics [29, 34]. All relevant quantities are known for independent, identically distributed (i.i.d.) variables. Much less is known about correlated variables.

II. PROBABILITY DISTRIBUTION FOR THE k^{th} LARGEST (I.I.D. VARIABLES)

As before, $p(y)$ is the parent distribution, and we denote the integrated parent as

$$\mu(z) = \int_{-\infty}^{z} dy p(y) \,. \tag{1}$$

N draws from the above distribution provides us with a batch of random variables $y_1, y_2, ..., y_N$. Let z be the k^{th} largest in the batch

$$y_1 \leq y_2 \leq ... \leq y_{k-1} \leq y_k = z \leq y_{k+1} \leq ... \leq y_N \,. \tag{2}$$

The probability distribution of the k-th maximum $P_N^{(k)}(z)$ is calculated from the condition that $k-1$ particles are above z, 1 particle is in the interval $[z, z+dz]$, while $k-1$ particles are above $z+dz$. The number of ways to distribute N particles with the above constraints is $N!/[(k-1)!1!(N-k)!]$ and so we have

$$P_N^{(k)}(z)dz = \frac{N!}{(k-1)!(N-k)!}\mu^{N-k}(z)p(z)dz[1-\mu(z)]^{k-1}. \tag{3}$$

The above expression can be simplified by using $p(z) = d\mu/dz$ and remembering that the distribution of the largest is given by $P_N^{(1)}(z) = N\mu^{N-1}d\mu/dz$:

$$P_N^{(k)}(z) = \frac{(N-1)...(N-k+1)}{(k-1)!}P_N^{(1)}(z)\left[\frac{1}{\mu(z)} - 1\right]^{k-1} . \tag{4}$$

Let us now make the appropriate change of variables $z = a_N x + b_N$. Then $P_1(z, N)$ converges to the limit distribution $P_N^{(1)}(z, N) \to P_1(x)$ and, since $M(a_N x + b_N, N) = \mu^N(a_N x + b_N) \to M(x)$, we can write

$$\begin{aligned} &\mu(z = a_N x + b_N) = M^{1/N}(a_N x + b_N, N) \to \\ &M^{1/N}(x) = 1 + \ln M(x)/N + O(1/N^2) \end{aligned} \tag{5}$$

and, consequently,

$$\left[\frac{1}{\mu(z)} - 1\right]^{k-1} \to \frac{[-\ln M(x)]^{k-1}}{N^{k-1}} \,. \tag{6}$$

Let us remember now that

$$M(x) = e^{-(1+\gamma x)^{-1/\gamma}} \quad ; \quad P_1(x) = dM(x)/dx \tag{7}$$

and use (6) and (7) in (4) to obtain the probability distribution of the position of the k^{th} largest

$$P_k(x) = \frac{1}{(k-1)!}(1+\gamma x)^{-k/\gamma - 1}e^{-(1+\gamma x)^{-1/\gamma}} \,. \tag{8}$$

In the limit of $\gamma \to 0$ (Gumbel class), the above expression converges to

$$P_k(x) = \frac{1}{(k-1)!}\exp\left[-kx - \exp(-x)\right] . \tag{9}$$

III. PICTURE GALLERY OF THE DISTRIBUTION OF THE k^{th} LARGEST

Figure 1 shows the distribution of the k^{th} largest for the Gumbel class ($\gamma = 0$) for various ks. As one can see, the distribution becomes narrower as k increases and, furthermore, it seems to approach a Gaussian distribution (a saddle-point calculation shows that $P_k(x)$ indeed converges to a Gaussian for large k).

One should note that these distribution functions emerged in the scaling limit. Thus, apart from the shift to infinity, on the real scale, they may collapse ($\delta > 0$) or the distances between the maxima may go to infinity ($\delta < 0$), or the distances between the maxima may become stationary [$p(y) \sim e^{-x}$] in the $N \to \infty$ limit.

Figure 2 displays $P_k(x)$ for the Frechet class ($\gamma > 0$) for various ks and for $\gamma = 1/5$, $1/2$ and 1. As one can see, the series of distribution functions for small $\gamma = 1/5$

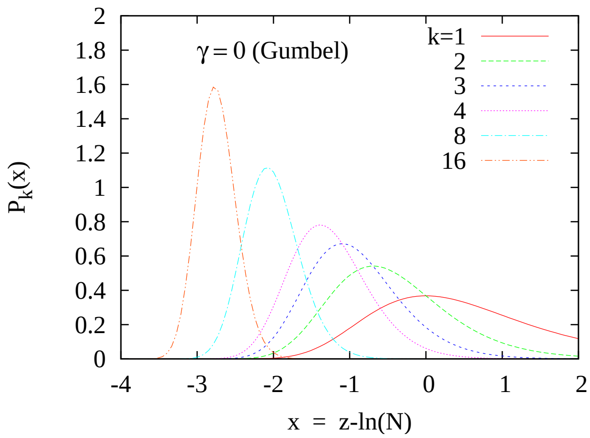

FIG. 1: Distribution of the k-th largest for a parent having the Gumbel as the limit distribution (e.g. $e^{-x^{\delta}}$).

is close to what we observe in case of the Gumbel class. For small γ, the Frechet, the Gumbel, and the Weibull classes are practically indistinguishable even if the full order statistics is considered.

A remarkable feature we can observe for increasing γ is that while the largest value has a wide spread, for $k = 8$ or 16, the k^{th} largest is much better localized. This is understandable. For a power law distribution, the outliers are far out but the majority of the draws are close to the average (it is worth to examine in detail the case of $p(y) \sim y^{-3}$).

Figure 3 shows the order statistics for the Weibull class ($\gamma < 0$). For this case we changed from maximum to minimum i.e. we are considering the parents which have a lower cutoff [$p(y) = 0$ for $y < y_0$] and the probability distribution of the k^{th} smalles $P_k(x)$ is displayed for $\gamma = -1/5$, $-1/2$, and -1.

We can see again that the small $\gamma = -1/5$ case is remarkably similar to what we obtained for the Gumbel class (apart from the direction of the x-axis due to studying the minimum insted of the maximum).

One should also note that for $\gamma = -1$ (the parent distribution is finite at the lower limit), the distribution of the smallest $P_1(x)$ has a maximum at $x = 0$ while the maximum is detached from zero for $k \geq 1$

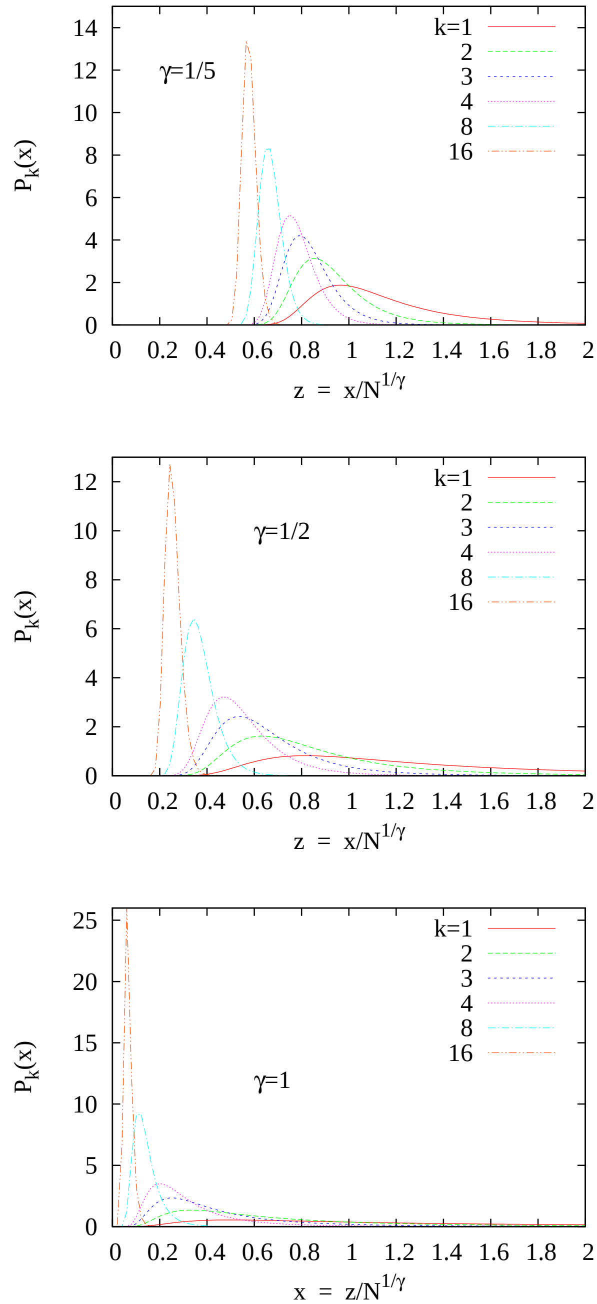

FIG. 2: Distribution of the k-th largest for a parent starting from $x = 1$ and having a large-x asymptote $P_s(x) \sim x^{-1/\gamma-1} \sim x^{-6}$, x^{-3}, x^{-2}, respectively.

IV. CALCULATION OF $\langle x \rangle_k$ FOR THE WEIBULL CLASS

We would like to see how does the spectrum of the average position of the k^{th} smallest $\langle x \rangle_k$ looks like. Let us consider the case when the minimum of the parent is at $y = 0$, and the small y asymptote of the parent is $p(y) \sim y^{-1/\gamma-1} = y^{1/|\gamma|-1}$. Then the distribution of the k^{th} smallest is given by

$$P_k(x) = \frac{1}{|\gamma|(k-1)!} x^{k/|\gamma|-1} \exp\left(-x^{1/|\gamma|}\right). \qquad (10)$$

The average $\langle x \rangle_k$ is now calculated as

$$\begin{aligned}\langle x \rangle_k &= \int_0^\infty \frac{x}{|\gamma|(k-1)!} x^{k/|\gamma|-1} \exp\left(-x^{1/|\gamma|}\right) dx \\ &= \frac{1}{(k-1)!}\int_0^\infty u^{k+|\gamma|-1} e^{-u} du = \frac{\Gamma(k+|\gamma|)}{\Gamma(k)}. \qquad (11)\end{aligned}$$

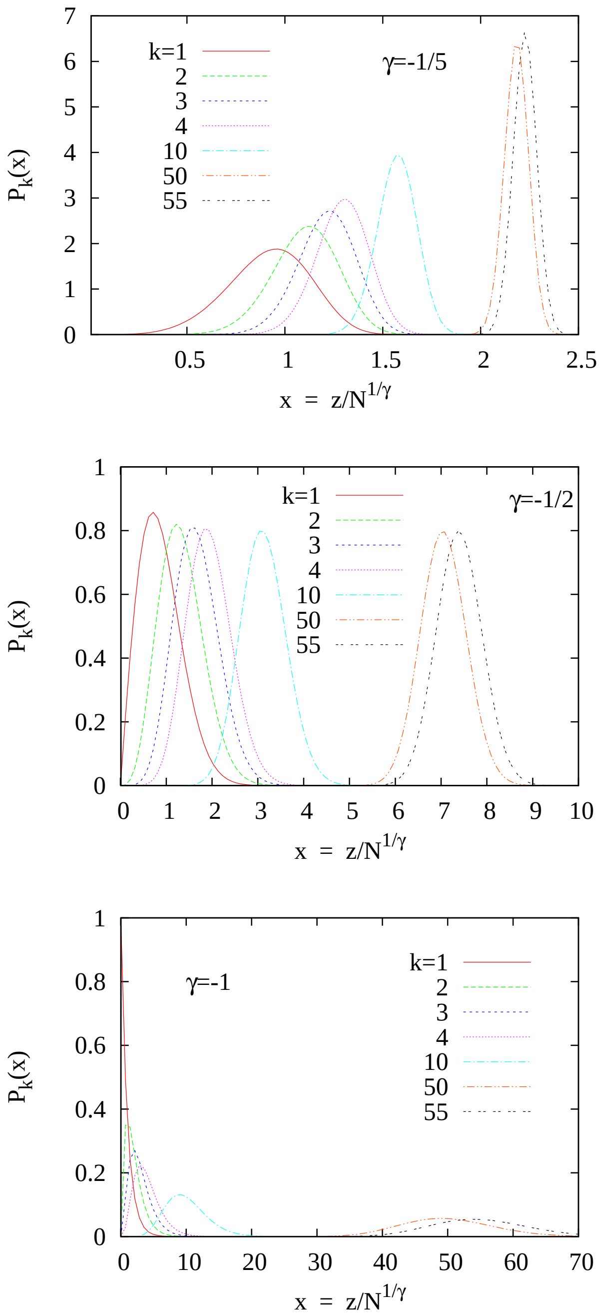

FIG. 3: Distribution of the k-th smallest for a parent starting from $x = 0$ and having a small-x asymptote $P_s(x) \sim x^{-1/\gamma-1} \sim x^4$, x, $const$, respectively.

Using now an identity valid for large k

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} \approx k^{a-b} \tag{12}$$

we obtain our final result from (11)

$$\langle x \rangle_k \sim k^{|\gamma|} \,. \tag{13}$$

It is remarkable that we obtained a discrete spectrum from the order statistics.

V. COMPARING WITH QUANTUM SPECTRA

We shall now compare the order statistics spectra to the energy spectra of quantum mechanical systems in the quasi-classical limit. The reason for this comparison, apart from its entertaining aspects, is that the discrete quantum mechanical spectra may also be considered as an order statistics spectra.

Let us consider a particle of mass m which moves in a potential

$$V(x) = g|x|^{\theta} \tag{14}$$

where $g > 0$ is the coupling constant. The simplest way to calculate the quasi-classical limit of the spectra is to use dimensional analysis combined with the observation that, in the large quantum-number limit ($k \to \infty$), the quantization condition ($\int pdq = kh$) forces h and k to appear in the combination hk. This means that the k-dependence of the spectra is determined by its h dependence. Since the energy is uniquely determined by the dimensions of m, q, and h, one obtains

$$E_k \sim (hk)^{2\theta/(\theta+2)} \,. \tag{15}$$

Comparing the above result with (13), one can see that there is a mapping between the quantum mechanical and the order statistics exponents:

$$|\gamma| = \frac{2\theta}{\theta+2} \,. \tag{16}$$

One can observe that the harmonic potential ($\theta = 2$) corresponds to a parent with $\gamma = -1$ i.e. to $p(y) \sim finite$ for $y \to 0$ while the square-well potential ($\theta \to \infty$) corresponds to $\gamma = -2$ (i.e. $p(y) \sim y^{-1/2}$).

It should be emphasized that it is not obvious that the mapping between the exponents have any physical content. Nevertheless, it is intriguing to ask whether there is a *quasi-classical* extreme value question whose answer is the quantum mechanical spectra.

VI. ASTROPHYSICAL APPLICATION: TREMAIN-RICHSTONE RATIOS

In astrophysics, the maximal luminosity of galaxies in a galaxy cluster has been investigated for a long time [36]). In general, the applicability of the iid. description of the EVS the maximum luminosities is doubted because the gap between the largest and the second largest appears to be bigger than the i.i.d. prediction. The notion is checked by considering some universal numbers which are coming from the given i.i.d. EVS distribution. Two simple quantities related to order statistics are the following ratios (introduced by Tremain and Richstone in

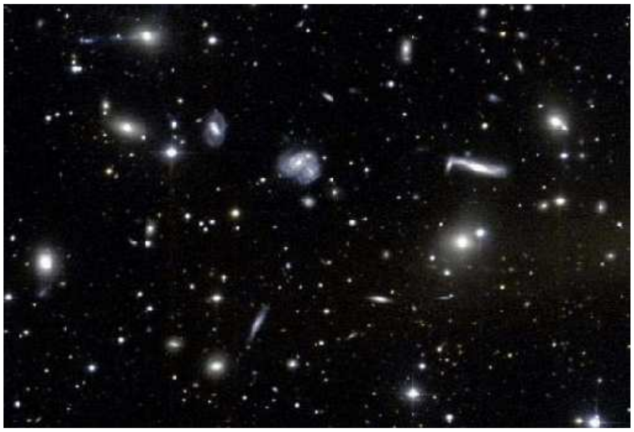

FIG. 4: Galaxy cluster Hercules.

1977 [36])

$$T_1 = \frac{\sigma_1}{d_{12}} \equiv \frac{\sqrt{\langle x_1^2\rangle - \langle x_1\rangle^2}}{\langle x_1 - x_2\rangle} \tag{17}$$

and

$$T_2 = \frac{\sigma_{12}}{d_{12}} \equiv \frac{\sqrt{\langle (x_1 - x_2)^2\rangle - \langle (x_1 - x_2)\rangle^2}}{\langle x_1 - x_2\rangle}\,. \tag{18}$$

The values of $T_1 = \pi/\sqrt{6} = 1.28$ and $T_2 = 1$ are quoted in the astrophysics literature for the Gumbel universality class. Furthermore, it is proved generally, that $T_1 \geq \pi/\sqrt{6}$ and $T_2 \geq 1$ for i.i.d. EVS statistics (for arbitrary γ). Below, we shall calculate T_1 and T_2 for the Gumbel class.

A. Calculation of T_1 for $\gamma = 0$

Practically, this has been done already in the previous lectures. Indeed, since T_1 is a dimensionless ratio of two differences, the a_N and b_N factors cancel when calculating it for large N and the limit distribution can be used to evaluate both the numerator and the denominator (of course, one should use the same standardization for the variable in the the limit distribution, e.g. $M(0) = M'(0) = 1/e$).

For $\gamma = 0$, using the Gumbel distribution, we find

$$\langle x_1\rangle = \int_{-\infty}^{\infty} dz z e^{-z-e^{-z}} = \gamma_E \tag{19}$$

where $\gamma_E \approx 0.577$ is the Euler constant and, furthermore, we obtain after some calculation or consulting integral tables

$$\langle x_1^2\rangle - \langle x_1\rangle^2 = \int_{-\infty}^{\infty} dz z^2 e^{-z-e^{-z}} - \gamma_E^2 = \frac{\pi^2}{6}\,. \tag{20}$$

The gap has been obtained for general γ [see eq.(15) in Lecture 3-4]

$$\langle x_1\rangle - \langle x_2\rangle = \gamma\langle x_1\rangle + 1\,. \tag{21}$$

Using the above result for $\gamma = 0$ and substituting (20) and (21) into (17), we obtain

$$T_1 = \frac{\sigma_1}{d_{12}} \equiv \frac{\sqrt{\langle x_1^2\rangle - \langle x_1\rangle^2}}{\langle x_1 - x_2\rangle} = \frac{\pi}{\sqrt{6}} \approx 1.283\,. \tag{22}$$

As one can see from Fig.5, T_1 measured from the largest

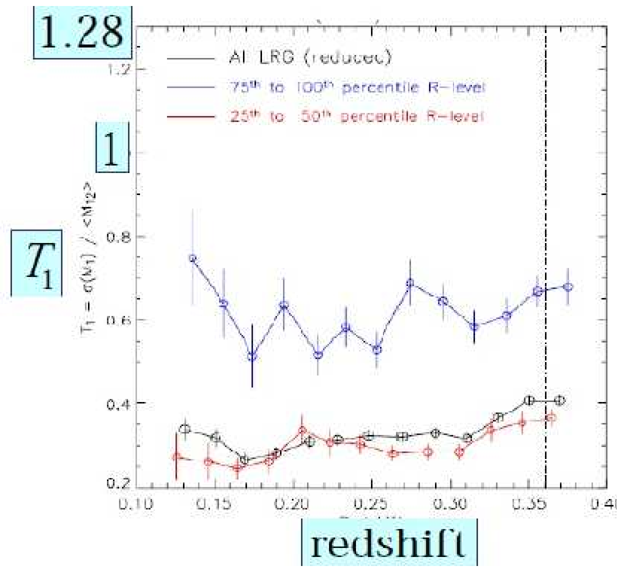

FIG. 5: Tremain ratio [T_1, see (17)] for galaxy clusters at various distances (redshifts). As one can see, the i.i.d. inequality ($T_1 \geq 1$) is trongly violated.

and second luminosities in galaxy clusters is significantly lower from the i.i.d. value $T_1 \approx 1.283$.

B. Calculation of T_2 for $\gamma = 0$, and the joint probability distribution of the largest and second largest

The evaluation of T_2 can also be done using any scale for the distribution function, so we shall again use the "$P(0) = 1/e$; $M(0) = 1/e$" convention where the distribution function of the largest and the second largest have the forms

$$P_1(x) = e^{-x-e^{-x}} \quad , \quad P_2(x) = e^{-2x-e^{-x}}\,. \tag{23}$$

The rescaled numerator in T_2 is the same as before $\langle (x_1 - x_2)\rangle = 1$, but in order to calculate the fluctuation of $x_1 - x_2$, we need to derive the joint distribution of $P(x_1, x_2)$.

The probability of z_1 being the largest, z_2 being the second largest (in the original variables), and all the others are smaller than z_2 is given by

$$P(z_1, z_2)dz_1 dz_2 = \tag{24}$$
$$N(N-1)\mu^{N-2}(z_2)\theta(z_1 - z_2)p(z_2)dz_2 p(z_1)dz_1$$

where $N(N-1)$ comes from combinatorics $[N!/(N-2)!1!1!]$, and $p(z) = d\mu/dz$ is the parent, while $\mu(z)$ is the integrated parent.

Making now the changes $z_i = a_N x_i + b_N$, and using the large N limit where

$$\mu^N(a_N x + b_N) \to M_1(x)\,, \tag{25}$$

and

$$\mu(a_N x + b_N) \approx M_1^{\frac{1}{N}}(x)\,, \tag{26}$$

we can write

$$\begin{aligned}
&P(x_1, x_2)\\
&= (N-1)\frac{dM_1(x_2)}{dx_2}\frac{p(a_N x_1 + b_N)}{\mu(a_N x_1 + b_N)}\theta(x_1 - x_2)\\
&= \frac{dM_1(x_2)}{dx_2}\frac{dM_1(x_1)}{dx_1}\frac{M_1^{\frac{1}{N}-1}(x_1)}{M_1^{\frac{1}{N}}(x_2)}\theta(x_1 - x_2)
\end{aligned}$$

For large N, we can write $M_1^{1/N}(x) = 1 + N^{-1}\ln M_1(x) \approx 1$ and, furthermore, using $[dM_1(x)/dx]/M_1(x) = d\ln M_1(x)/dx$, we obtain

$$\begin{aligned}
&P(x_1, x_2) = \theta(x_1 - x_2)P_1(x_2)\frac{\ln M_1(x_1)}{dx_1} = \\
&\theta(x_1 - x_2)(1+\gamma x_1)^{-\frac{1}{\gamma}-1}(1+\gamma x_2)^{-\frac{1}{\gamma}-1}e^{-(1+\gamma x_2)^{-\frac{1}{\gamma}}}
\end{aligned} \tag{27}$$

The above distribution function becomes especially simple for the FTG class ($\gamma \to 0$), where we have

$$P(x_1, x_2) = \theta(x_1 - x_2)e^{-e^{-x_2}}e^{-x_1}\,. \tag{28}$$

This expression can actually be interpreted in the following way: x_2 is the maximum (out of $N-1 \to \infty$) and $x_1 > x_2$, with the distribution x_1 becoming exponential due to rescaling.

We are ready now to calculate $\langle (x_1 - x_2)^2 \rangle$ needed for the second Treamain-Richstone ratio. We shall calculate separately the terms in

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle - 2\langle x_1 x_2 \rangle + \langle x_2^2 \rangle\,. \tag{29}$$

Let us begin by showing that $\langle x_2^2 \rangle$ can be expressed through $\langle x_1 \rangle$ and $\langle x_1^2 \rangle$

$$\begin{aligned}
\langle x_2^2 \rangle &= \int_{-\infty}^{\infty} dz z^2 e^{-2z-e^{-z}} = \int_{-\infty}^{\infty} dz z^2 e^{-z}\tfrac{d}{dz}e^{-e^{-z}}\\
&= z^2 e^{-z}e^{-e^{-z}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dz(2z - z^2)e^{-z-e^{-z}}\\
&= \langle x_1^2 \rangle - 2\langle x_1 \rangle\,.
\end{aligned} \tag{30}$$

Next, we calculate the correlation ($\langle x_1 x_2 \rangle$) part. Using the joint distribution (28), we have

$$\begin{aligned}
\langle x_1 x_2 \rangle &= \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{\infty} dz_1 z_1 z_2 \theta(z_1 - z_2)e^{-z_2-e^{-z_2}}e^{-z_1}\\
&= \int_{-\infty}^{\infty} dz_2 z_2 e^{-z_2-e^{-z_2}} \int_{z_2}^{\infty} dz_1 z_1 e^{-z_1}\\
&= \int_{-\infty}^{\infty} dz_2 z_2 e^{-z_2-e^{-z_2}}(z_2+1)e^{-z_2}\\
&= \int_{-\infty}^{\infty} dz_2 (z_2^2 + z_2)e^{-2z_2-e^{-z_2}} = \langle x_2^2 \rangle + \langle x_2 \rangle\,.
\end{aligned} \tag{31}$$

Collecting the terms (30) and (31), we obtain

$$\begin{aligned}
\langle (x_1 - x_2)^2 \rangle &= \langle x_1^2 \rangle + \langle x_1^2 \rangle - 2\langle x_1 \rangle - 2(\langle x_2^2 \rangle + \langle x_2 \rangle)\\
&= 2\langle x_1^2 \rangle - 2\langle x_1 \rangle - 2(\langle x_1^2 \rangle - 2\langle x_1 \rangle + \langle x_2 \rangle)\\
&= 2\langle x_1 \rangle - 2\langle x_2 \rangle = 2
\end{aligned} \tag{32}$$

Earlier (21) we found $\langle (x_1 - x_2) \rangle = 1$. So, the fluctuation of the gap between the 1^{st} and the 2^{nd} largest is obtained as

$$\langle (x_1 - x_2)^2 \rangle - \langle (x_1 - x_2) \rangle^2 = 1\,. \tag{33}$$

and consequently, the second Tremain ratio is obtained in agreement with calculations [36] for the exponential parent distribution belonging to the FTG class

$$T_2 = \frac{\sqrt{\langle (x_1 - x_2)^2 \rangle - \langle (x_1 - x_2) \rangle^2}}{\langle x_1 - x_2 \rangle} = 1 \tag{34}$$

The measurements do not agree with the i.i.d. values of T_1 and T_2. This suggests strong correlations among the galaxies in a cluster. Before concluding this, however, one must examine what are the effects coming from the finite number of galaxies in a cluster (e.g. $N = 100$ in the Hercules cluster shown in Fig.4). Thus we arrive to the problem of finite-size corrections.

VII. HOMEWORKS

Homework 7: Consider the photons in a room of temperature T. Determine the average energy of the photon with the largest energy. Find the avarage difference between the largest and second largest energies of the photons.

Homework 8: Calculate the 1^{st} Tremain-Richstone ratio T_1 for $-\infty < \gamma \leq 1/2$ (note that the second moments of the limit distribution will have to be calculated and it does not exist for $\gamma \geq 1/2$). Show that $T_1(\gamma) > 1$ in the allowed range of γs.

[1] R. A. Fisher and L. H. C. Tippett, Procs. Cambridge Philos. Soc. **24**, 180 (1928).
[2] J. Galambos, The Asymptotic Theory of Extreme Value Statistics (John Wiley & Sons, New York, 1978).
[3] L. de Haan and A. Ferreira, Extreme Value Theory: An Introduction (Springer, New York, 2006).
[4] E. J. Gumbel, *Statistics of Extremes* (Dover Publications, 1958).
[5] W. Weibull, J. Appl. Mech.-Trans. ASME **18**, 293 (1951).
[6] (1997) P. Embrecht, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance* (Springer, Berlin, 1997).
[7] R. W. Katz, M. B. Parlange, and P. Naveau, Adv. Water Resour. **25**, 1287 (2002).
[8] H. v. Storch and F. W. Zwiers, *Statistical Analysis in Climate Research* (Cambridge University Press, Cambridge, 2002).
[9] J. F. Eichner, J. W. Kantelhardt, A. Bunde, and S. Havlin, Phys. Rev. E **73**, 016130 (2006).
[10] B. Gutenberg and C. F. Richter, Bull. Seismol. Soc. Am. **34**, 185 (1944).
[11] M. J. Geller and P. J. E. Peebles, ApJ **206**, 939 (1976).
[12] Y.-T. Lin, J. P. Ostriker, and C. J. Miller, ApJ **715**, 1486 (2010).
[13] J.-P. Bouchaud and M. Mézard, J. Phys. A **30**, 7997 (1997).
[14] P. L. Krapivsky and S. N. Majumdar, Phys. Rev. Lett. 85, 5492 (2000).
[15] M. Z. Bazant, Phys. Rev. E **62**, 1660 (2000).
[16] A. Lakshminarayan, S. Tomsovic, O. Bohigas, and S. N. Majumdar, Phys. Rev. Lett. **100**, 044103 (2008).
[17] S. Raychaudhuri, M. Cranston, C. Przybyla, and Y. Shapir, Phys. Rev. Lett. **87**, 136101 (2001).
[18] G. Györgyi, P. C. W. Holdsworth, B. Portelli, and Z. Rácz, Phys. Rev. E **68**, 056116 (2003).
[19] S. N. Majumdar and A. Comtet, Phys. Rev. Lett. 92, 225501 (2004).
[20] H. Guclu and G. Korniss, Phys. Rev. E **69**, 065104(R) (2004).
[21] C. J. Bolech and A. Rosso, Phys. Rev. Lett. **93**, 125701 (2004).
[22] D.-S. Lee, Phys. Rev. Lett. **95**, 150601 (2005).
[23] G. Schehr and S. N. Majumdar, Phys. Rev. E **73**, 056103 (2006).
[24] G. Györgyi, N. R. Moloney, K. Ozogány, and Z. Rácz, Phys. Rev. E **75**, 021123 (2007).
[25] S. Sabhapandit and S. N. Majumdar, Phys. Rev. Lett. **98**, 140201 (2007).
[26] T.W. Burkhardt, G. Györgyi, N. R. Moloney, and Z. Rácz, Phys. Rev. E **76**, 041119 (2007).
[27] S. N. Majumdar and A. J. Bray, Phys. Rev. Lett. **86**, 3700 (2001).
[28] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, 2001)
[29] M. R. Leadbetter, G. Lindgren, and H. Rootzen, *Extremes and Related Properties of Random Sequences and Processes* (Springer-Verlag, New York, 1982).
[30] S. N. Majumdar, Curr. Sci. **77**, 370 (1999).
[31] S. Redner and M. R. Petersen, Phys. Rev. E **74**, 061114 (2006).
[32] G. Wergen and J. Krug, EPL, 30008 (2010).
[33] G. Wergen, M. Bogner, and J. Krug, Phys. Rev. E **83**, 051109 (2011).
[34] J. Pickands, Annals of Statistics **3**, 119 (1975).
[35] L. Dobos and I. Csabai, MNRAS **414**, 1862 (2011)
[36] S. Tremaine, D. Richstone, ApJ **212** 311 (1977).
[37] E. Brunet and B. Derrida, EPL **87**, 60010 (2009).
[38] Y. V. Fyodorov, P. Le Doussal, and A. Rosso, J. Stat. Mech.: Theory and Exp., P10005 (2009).
[39] T. Antal, M. Droz, G. Györgyi, and Z. Rácz, Phys. Rev. E **65**, 046140 (2002).
[40] G. Györgyi, N. R. Moloney, K. Ozogány, and Z. Rácz, Phys. Rev. Lett. **100**, 210601 (2008).
[41] S. M. Berman, Ann. Math. Stat. 33, 502 (1964).
[42] J. F. Eichner, E. Koscielny-Bunde, A. Bunde, S. Havlin, and H.-J. Schellnhuber, Phys. Rev. E 68, 046133 (2003)