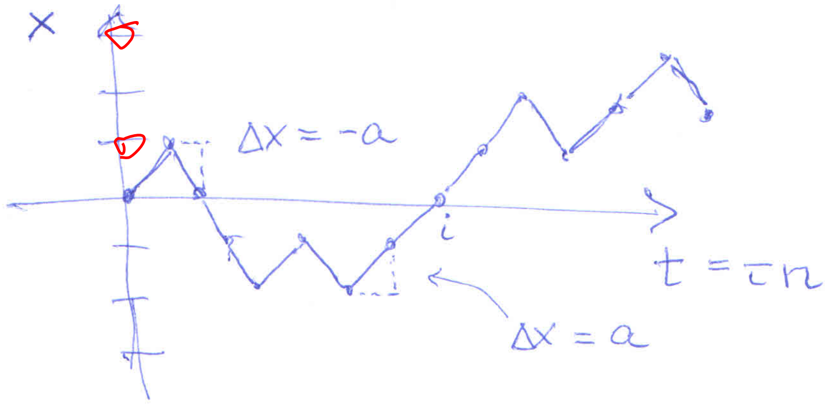
Correlations of the random walk positions



independent $\pm a$ steps

$$\Delta x_i = a e_i \qquad e_i = \pm 1 \qquad P(e_i = 1) = P(e_i = -1) = \frac{1}{2}$$

Although the steps are independent, the positions are correlated!

Calculate $\langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = C(i,j)$

$$x_i = \sum_{k=1}^{i} a e_k \quad \rightarrow \quad \langle x_i \rangle = a \sum_{k=1}^{i} \langle e_k \rangle = 0$$

$$\langle x_i^2 \rangle = a^2 \sum_{k=1}^{i} \sum_{l=1}^{i} \langle e_k e_l \rangle = a^2 \sum_{k=1}^{i} \langle e_k^2 \rangle = a^2 i$$

$$\sqrt{\langle x_i \rangle^2} = a \sqrt{i}$$

$$\langle x_i x_j \rangle = a^2 \sum_{k=1}^{i} \sum_{l=1}^{j} \langle e_k e_l \rangle = a^2 \min(i,j)$$

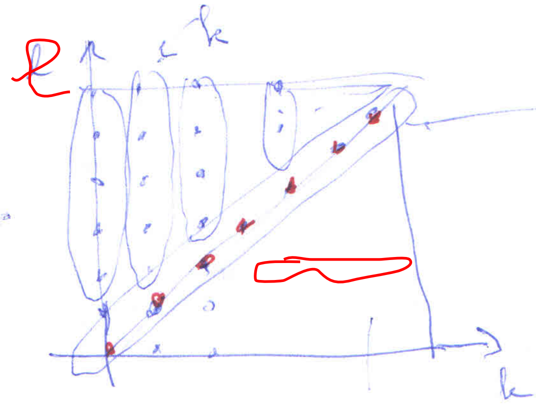
$$i < j \quad \sum_{k=1}^{i} \sum_{l=1}^{\cancel{j}=i} \langle e_k e_l \rangle = \sum_{k=1}^{i} \langle e_k^2 \rangle = i$$

$$i > j \quad \sum_{k=1}^{\cancel{i}=j} \sum_{l=1}^{j} \langle e_k e_l \rangle = \sum_{k=1}^{j} \langle e_k^2 \rangle = j$$

Calculation of the fluctuations of the "extensive" quantity

$$X = \sum_{k=1}^{N} x_i \qquad \langle X \rangle = \sum_{k=1}^{N} \langle x_i \rangle = 0$$

$$\langle X^2 \rangle = \sum_{k=1}^{N}\sum_{l=1}^{N} \langle x_k x_l \rangle = a^2 \sum_{k=1}^{N}\sum_{l=1}^{N} \min(k,l)$$



$$a^2 \sum_{k=1}^{N}\sum_{l=k}^{N} k\,\delta_{lk} = a^2 \frac{N(N+1)}{2}$$

$$a^2 \sum_{k=1}^{N} k\,(N-k) = a^2 \left[\frac{N(N+1)}{2} N - \frac{N(N+1)(2N+1)}{6} \right] =$$

$$= a^2 \frac{N(N+1)(N-1)}{6} \simeq a^2 \frac{N^3}{6} \quad \times 2 \quad k>l$$

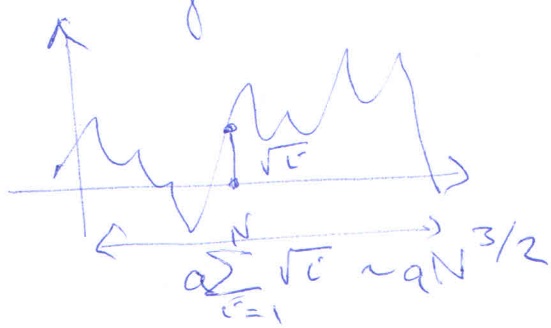
$$\langle X^2 \rangle \simeq a^2 \frac{N^3}{3}$$

$$\sqrt{\langle X^2 \rangle} \simeq \frac{a}{\sqrt{3}} N^{3/2}$$

Strong correlation case!

Some understanding

$$\sqrt{\langle x_i x_i \rangle} = a\sqrt{i}$$



Extreme value statistics of random walks (RW)

Q: RW starts at x_0

What is the probability that the maximum value $x_{max}(t)$ is less than z

$$x_{max}(t) = \max_t \left[x(t) - x_0 \right]$$

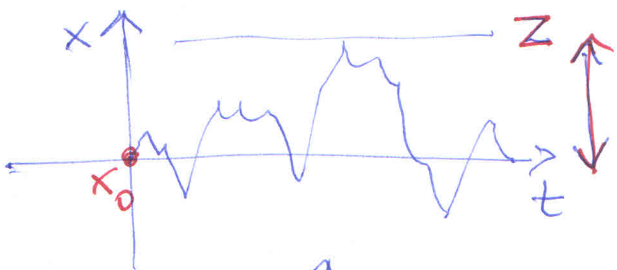
Integrated probability:

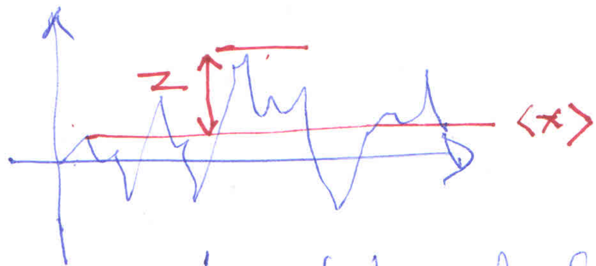
$$M(x<z\,|\,x_0;t) = ?$$

Probability density:

$$P(z\,|\,x_0;t) = \frac{dM(x<z\,|\,x_0;t)}{dz} = ?$$

Note: there are various definitions





this is what we are dealing with

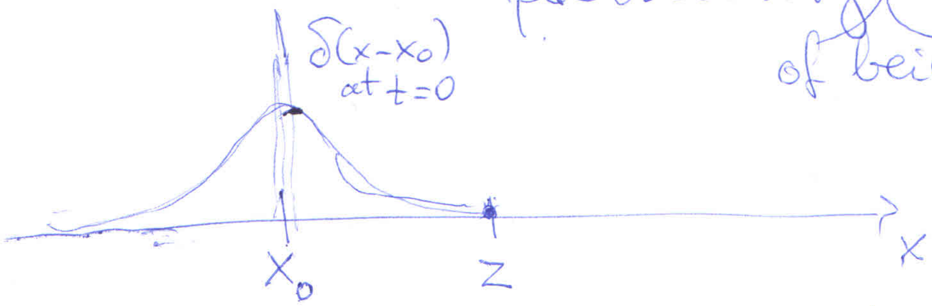
hard to calculat

$M(x<z\,|\,x_0;t)$ is calculated using the identity

$$M(x<z\,|\,x_0;t) = \int_{-\infty}^{z} \hat{P}_z(x,t\,|\,x_0,0)\,dx \qquad *$$

Probability that, starting from x_0 at $t=0$, the path of the random walker did not cross $x=z$ by the time t.

$\hat{P}_z(x,t|x_0,0)$ is a random walk, whose probability of being at $x=z$ is 0.



We have to solve the diffusion equation with the initial condition

$$\hat{P}_z(x,t=0|x_0,0) = \delta(x-x_0)$$

and with the boundary condition (BC)

$$\hat{P}_z(z,t|x_0,0) = 0$$

We know how to solve the diff. equation:

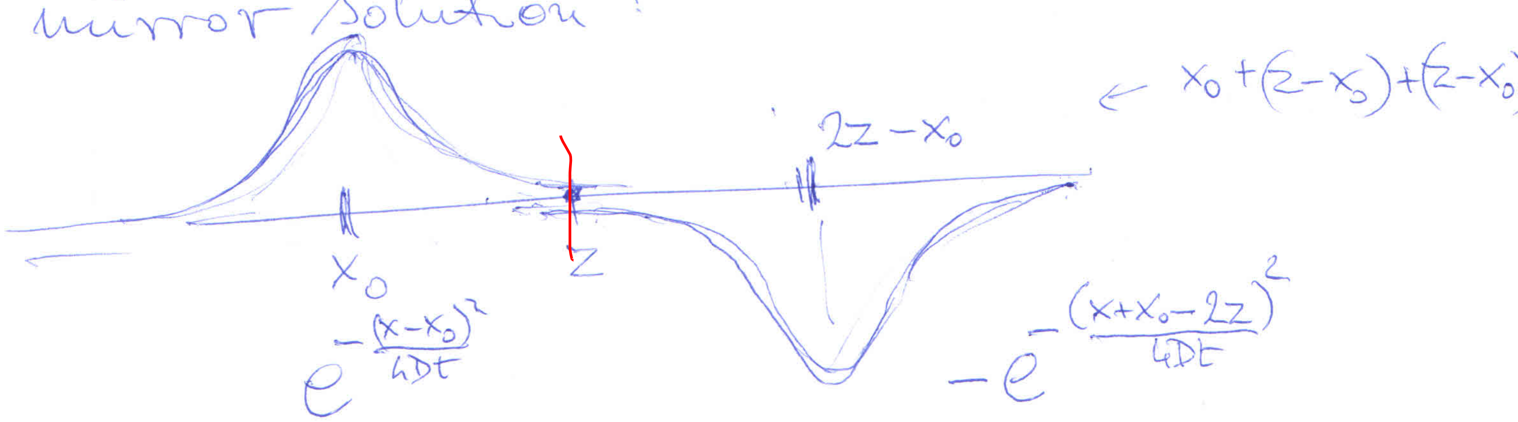
$$\partial_t \hat{P}_z(x,t|x_0,0) = D\,\partial_x^2 \hat{P}_z(x,t|x_0,0)$$

$$\downarrow$$

Without BC.

$$\hat{P}_z(x,t|x_0,0) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{(x-x_0)^2}{4Dt}}$$

The BC can be satisfied by adding a mirror solution:



Solution:

$$\hat{P}_z(x,t|x_0,0) = \frac{1}{\sqrt{4\pi D t}}\left[e^{-\frac{(x-x_0)^2}{4Dt}} - e^{-\frac{(x+x_0-2z)^2}{4Dt}} \right]$$

Using $*$:

$=0$ BC!

$$P(z|x_0;t) = \frac{\partial M(x<z|x_0,t)}{\partial z} = \hat{P}_z(x,t|x_0,0)\Big|_{-\infty}^{z} + \int_{-\infty}^{z} \frac{\partial \hat{P}_z(x,t|x_0,0)}{\partial z}\, dx$$

$$\frac{\partial \hat{P}_z(x,t|x_0,0)}{\partial z} = \frac{2}{\sqrt{\pi D t}} \frac{\partial}{\partial x} e^{-\frac{(x+x_0-2z)^2}{4Dt}}$$

$$\boxed{P(z|x_0;t) = \frac{1}{\sqrt{\pi D t}} e^{-\frac{(x_0-z)^2}{4Dt}}}$$

$z \geq x_0$

normalized