# Order statistics in $1/f^{\alpha}$ signals

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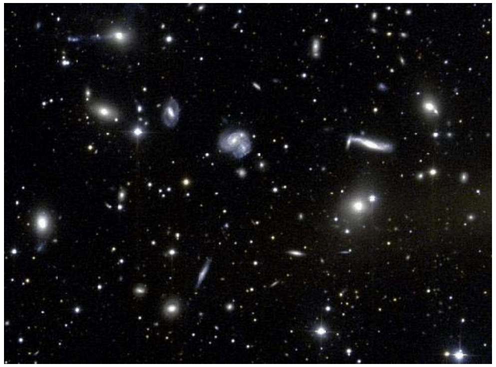
**Order statistics:**

What is it and why is it interesting?

Astrophysical examples:



**Importance of correlations:**

$1/f^{\alpha}$ process as a simple model system.

**Simulations:**

Emergence of power law spectra

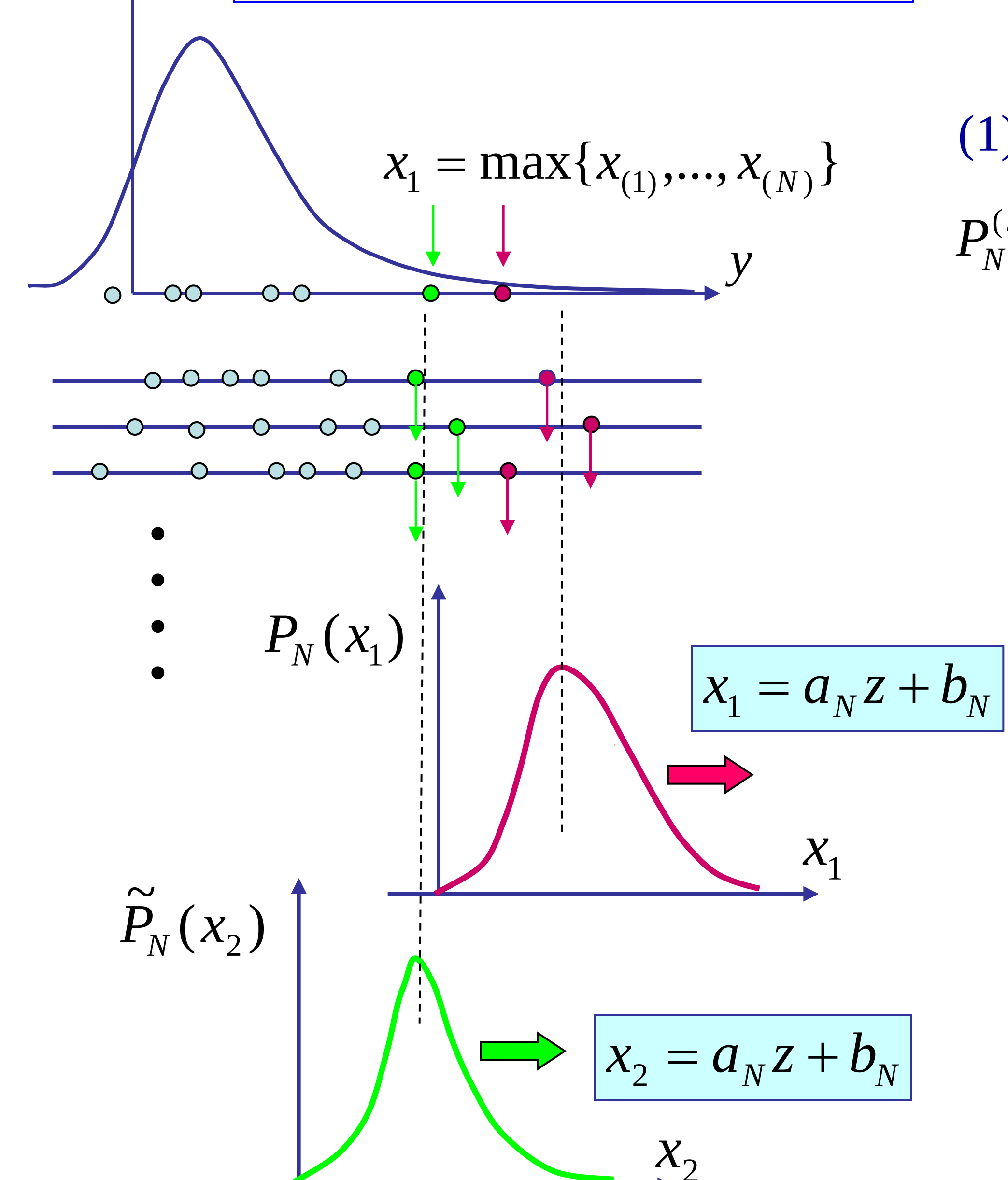
Scaling regimes: $0 \le \alpha < 1 \;\; ; \;\; 1 < \alpha \le 5 \;\; ; \;\; 5 \le \alpha < \infty$

**Phenomenology:**

Scaling arguments and derivation of the exponents

Memoriter: Comparison with quantum spectra

# Order statistics (iid)




**What are we interested in?**

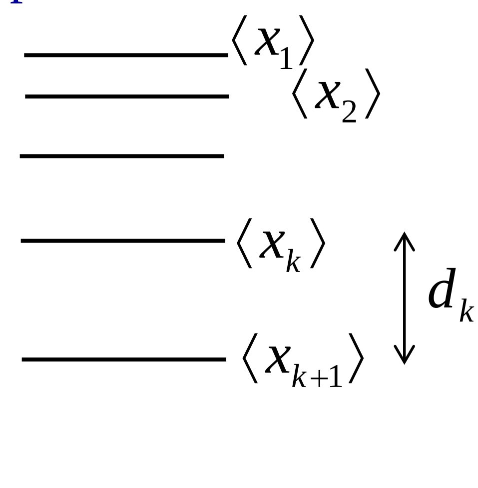
(1) Limit distributions

$$P_N^{(k)}(x_k = a_N z + b_N) \xrightarrow[N\to\infty]{} \Phi_k(z)$$

(2) Gap (scaled or unscaled)

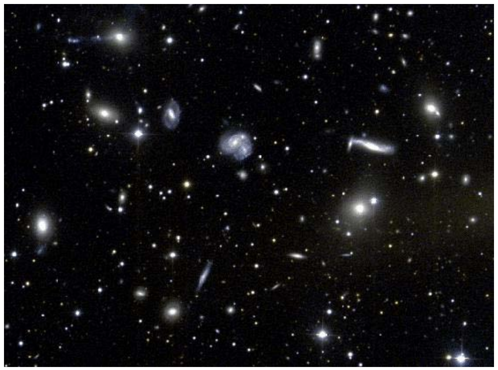
$$d_1 = \langle x_1 - x_2 \rangle$$

(3) Spectrum



# Motivation: Astrophysical problems

**Standard candles:** Using the brightest or the 2nd brightest, or the 3rd …?

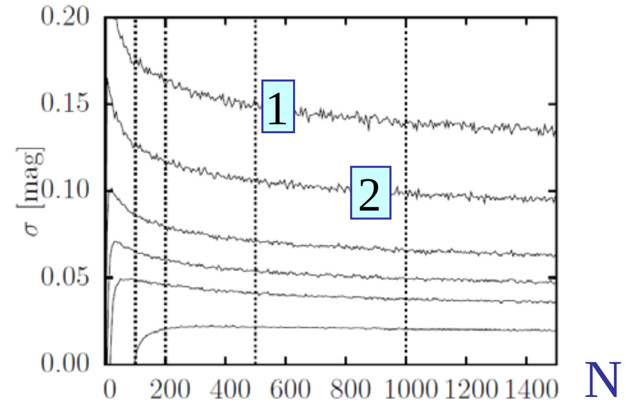


N ~ 100

Hercules Cluster

Dobos-Csabai, 2011

St. dev. of magnitude

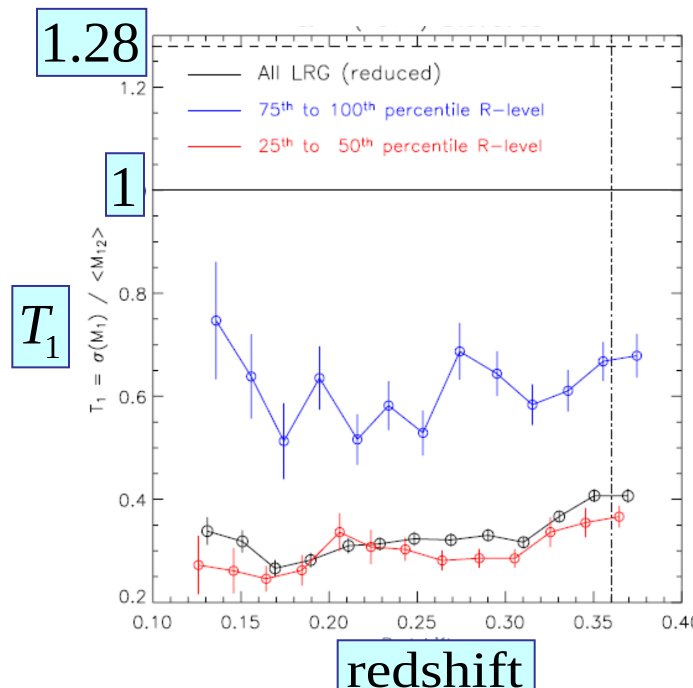



**Testing for iid:**

Tremain-Richstone, 1977

$$T_1 = \frac{\sqrt{\langle x_1^2 \rangle - \langle x_1 \rangle^2}}{\langle x_1 - x_2 \rangle} \geq 1$$

iid Gumbel class: $T_1 = 1.28$




Loh
-Strauss
2006

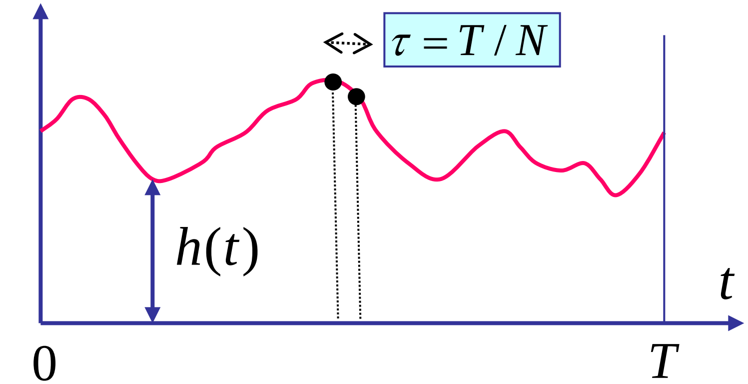
# Gaussian $1/f^{\alpha}$ signals

$N$ independent, nonidentically distributed Fourier modes

$$P(\{h_q\}) \sim e^{-\sigma_T \sum_q^{(N)} |q|^{\alpha} |h_q|^2}$$

with singular fluctuations

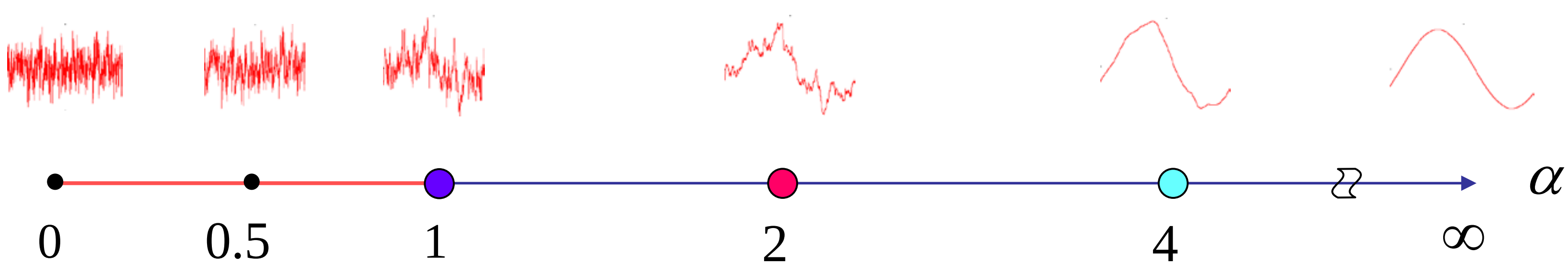
$$\langle |h_q|^2 \rangle \sim q^{-\alpha}$$




$$\langle w^2 \rangle = \overline{(h-\bar{h})^2} = T^{-1} \sum_q \langle |h_q|^2 \rangle \sim T^{\alpha-1}$$

**Edwards-Wilkinson**

**Mullins-Herring**




| **White noise** | $1/f$ **noise** | **Random walk** | **Random acceleration** | **Single mode, random phase** |
|---|---|---|---|---|

**Berman, 1964**

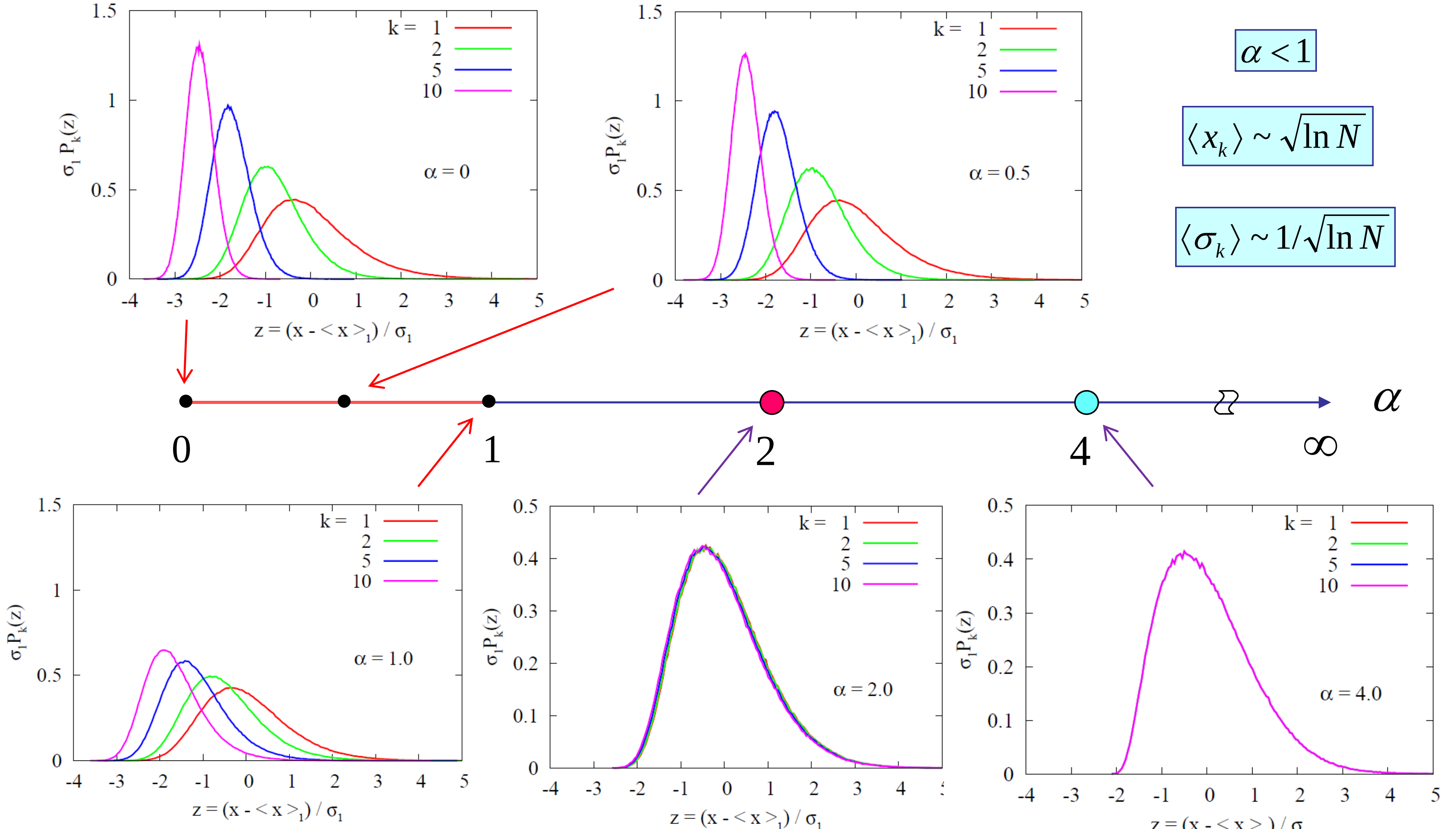
**EVS**

**Fyodorov et al., 2009**

**Majumdar-Comtet, 2004**

**T. Burkhardt et al., 2007**

# Distribution of $1^{st}$, $2^{nd}$, … , $10^{th}$




$\alpha < 1$

$\langle x_k \rangle \sim \sqrt{\ln N}$

$\langle \sigma_k \rangle \sim 1/\sqrt{\ln N}$

$\langle x_k \rangle \sim \langle \sigma_k \rangle \sim N^{(\alpha-1)/2}$

$\alpha > 1$

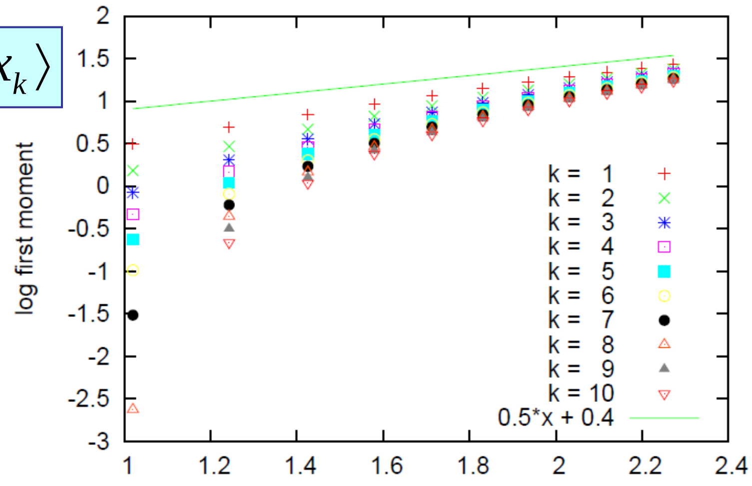
**Györgyi et al. 2007**

# Averages of 1st , 2nd , … , 10th

System sizes: $N = 16, 32, \ldots, 16384$ Number of runs: $1\,000\,000$

$\alpha = 0.5$

$\ln\langle x_k \rangle$

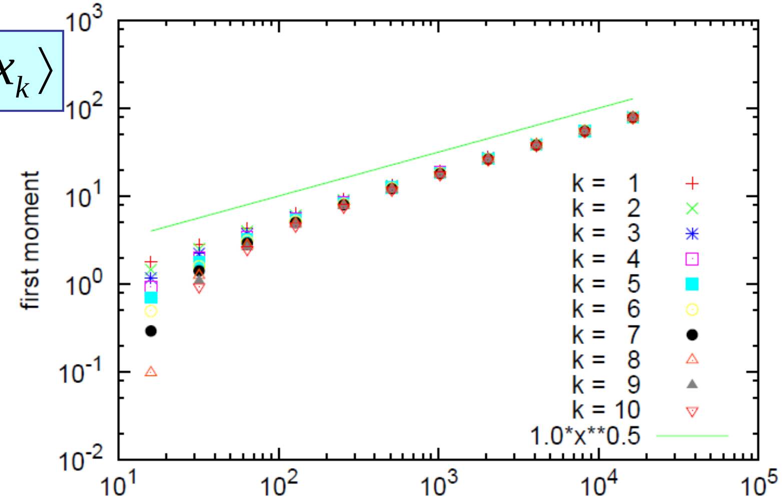



$\ln \ln N$

$\langle x_k \rangle \sim \sqrt{\ln N}$

$\alpha = 2$

$\langle x_k \rangle$

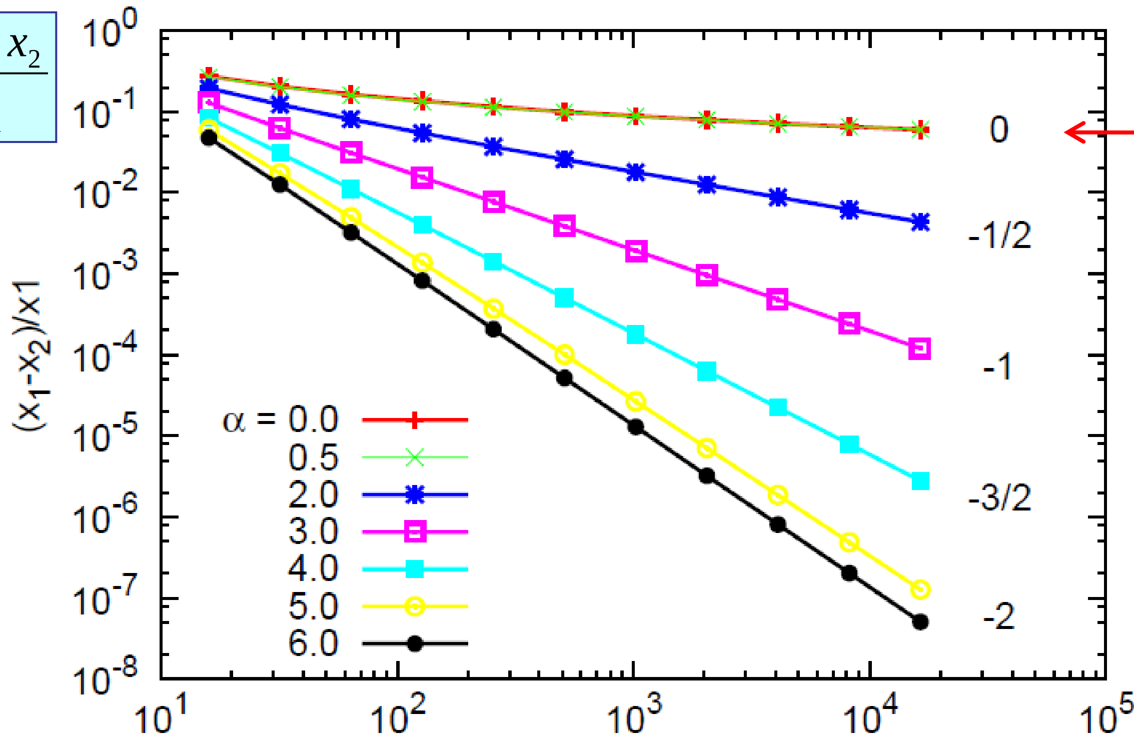



$N$

$\langle x_k \rangle \sim \mathrm{N}^{1/2}$

# Gap (scaled): I

$$\frac{x_1 - x_2}{x_1}$$



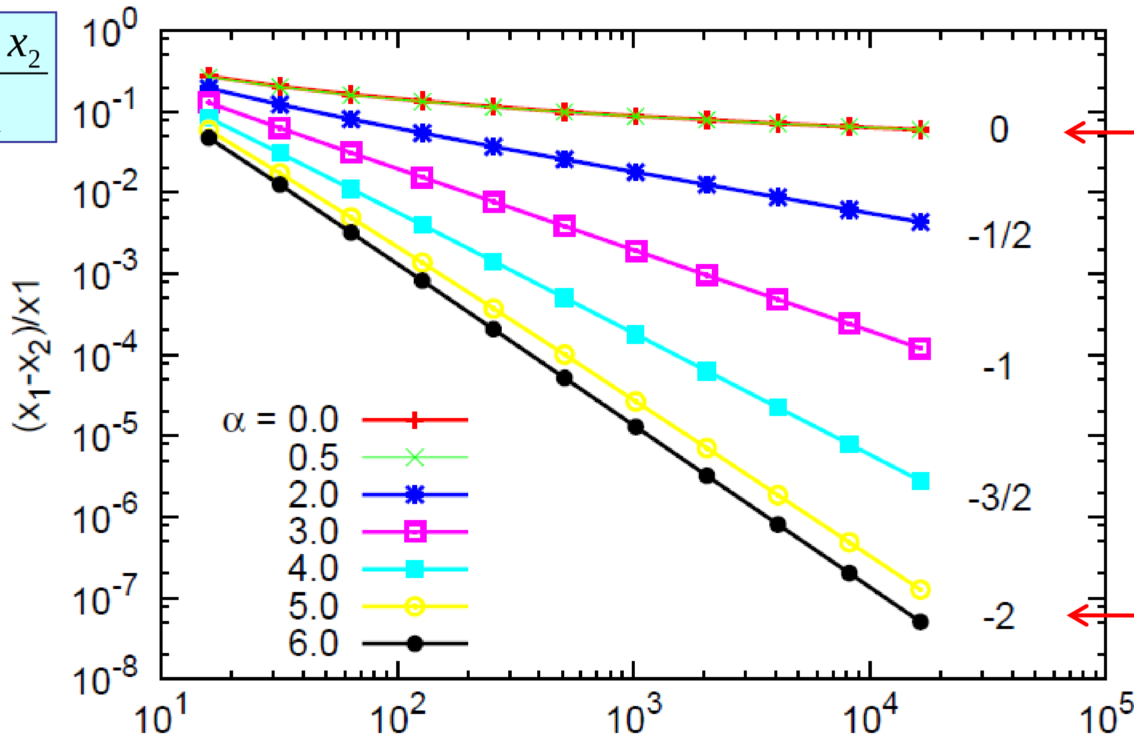

$N$

$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{\ln N}$$

$$\alpha = 0 \quad \rightarrow \quad x_1 \approx \frac{1}{\sqrt{\ln N}} z + \sqrt{\ln N} \quad \rightarrow \quad x_1 \approx \sqrt{\ln N} \quad \rightarrow \quad x_1 - x_2 \approx \frac{1}{\sqrt{\ln N}}$$

# Gap (scaled): II

$$\frac{x_1 - x_2}{x_1}$$



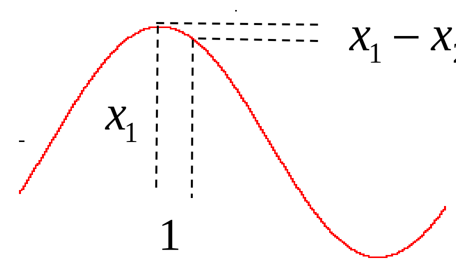

$N$

$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{\ln N}$$

$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{N^2}$$

$$\frac{N^{(\alpha-5)/2}}{N^{(\alpha-1)/2}}$$

$\alpha = \infty$

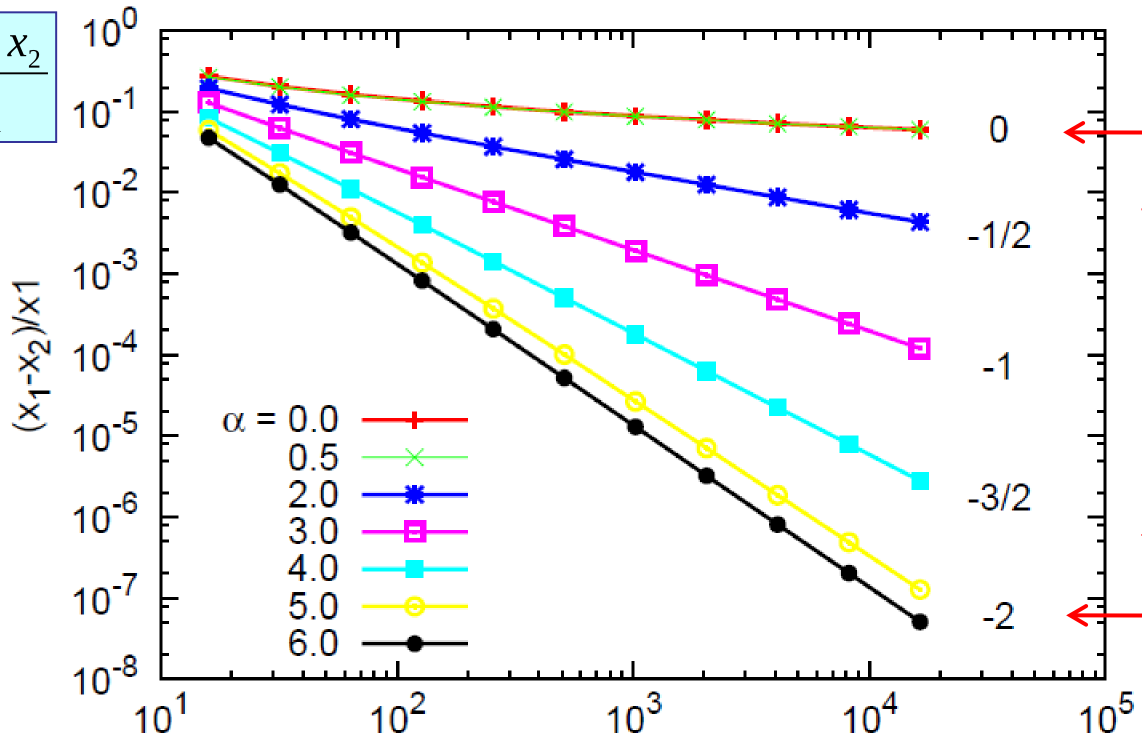



$x_1 - x_2 \approx x_1 / N^2$

$\alpha \geq 5$ → 2nd derivative exists

# Gap (scaled): III

$$\frac{x_1 - x_2}{x_1}$$




$N$

$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{\ln N}$$ $0 \le \alpha < 1$

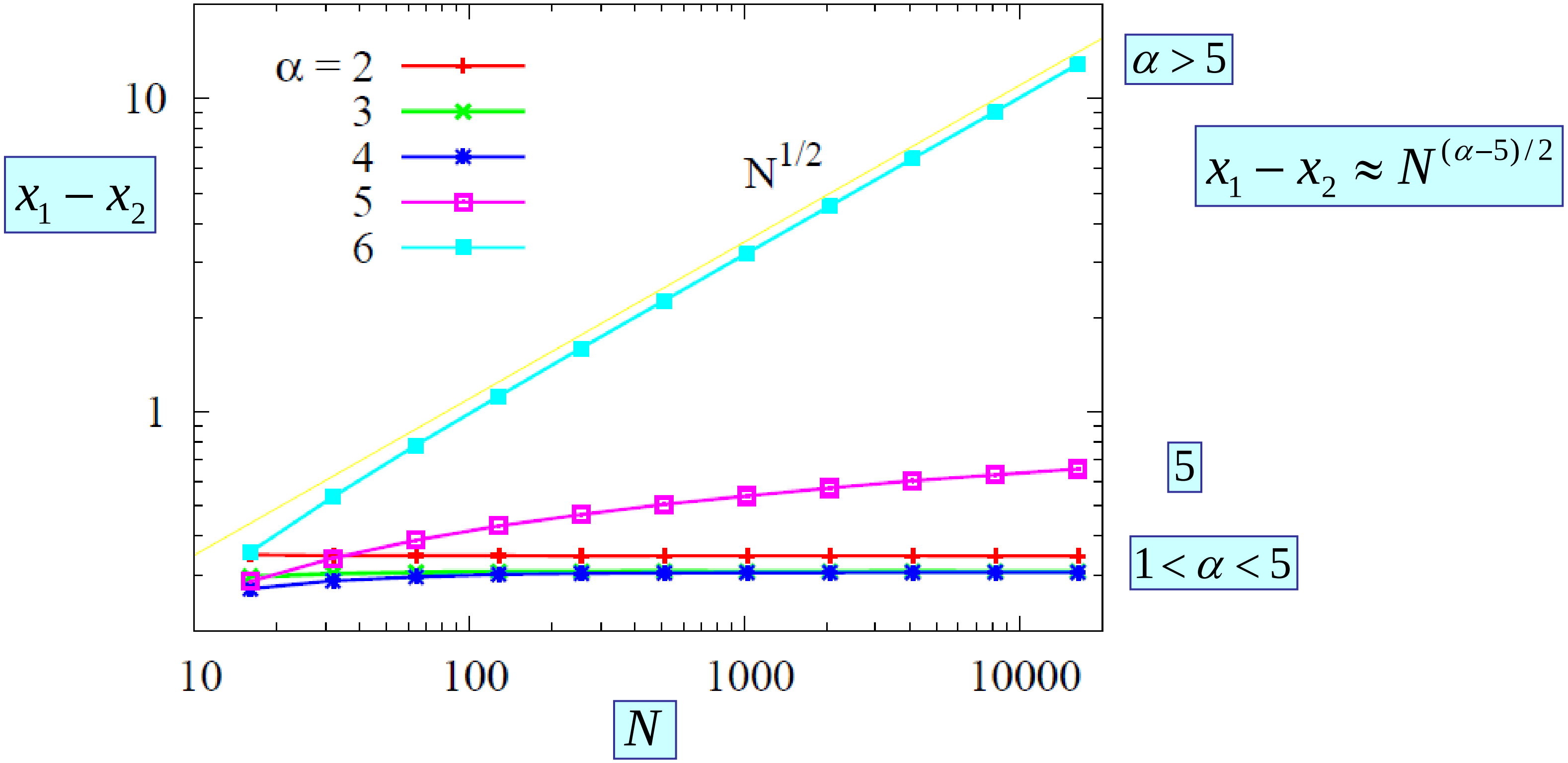
$$\frac{x_1 - x_2}{x_1} \approx \frac{\text{const}}{N^{(\alpha-1)/2}}$$ $1 < \alpha < 5$

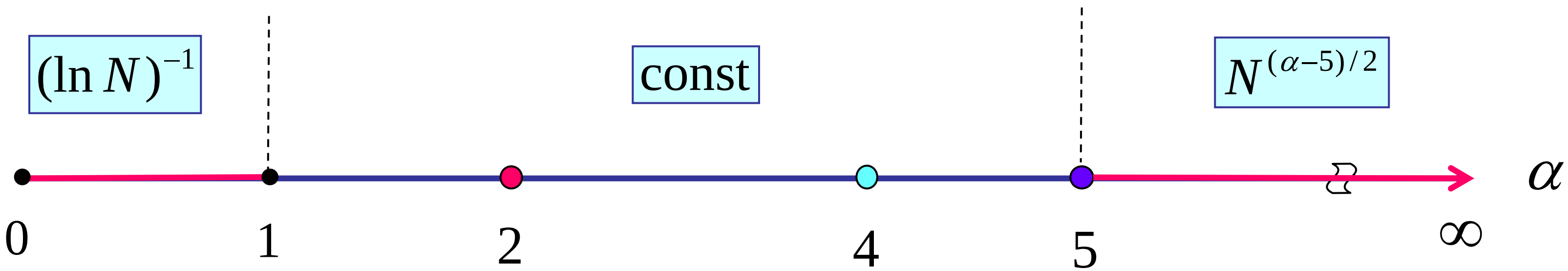
$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{N^2}$$ $5 < \alpha < \infty$

$$\frac{N^{(\alpha-5)/2}}{N^{(\alpha-1)/2}}$$

$1 < \alpha \le 5$ → $x_1 \approx N^{(\alpha-1)/2}$ and $\frac{x_1 - x_2}{x_1} \approx \frac{1}{N^{(\alpha-1)/2}}$ → $x_1 - x_2 \approx \text{const}$

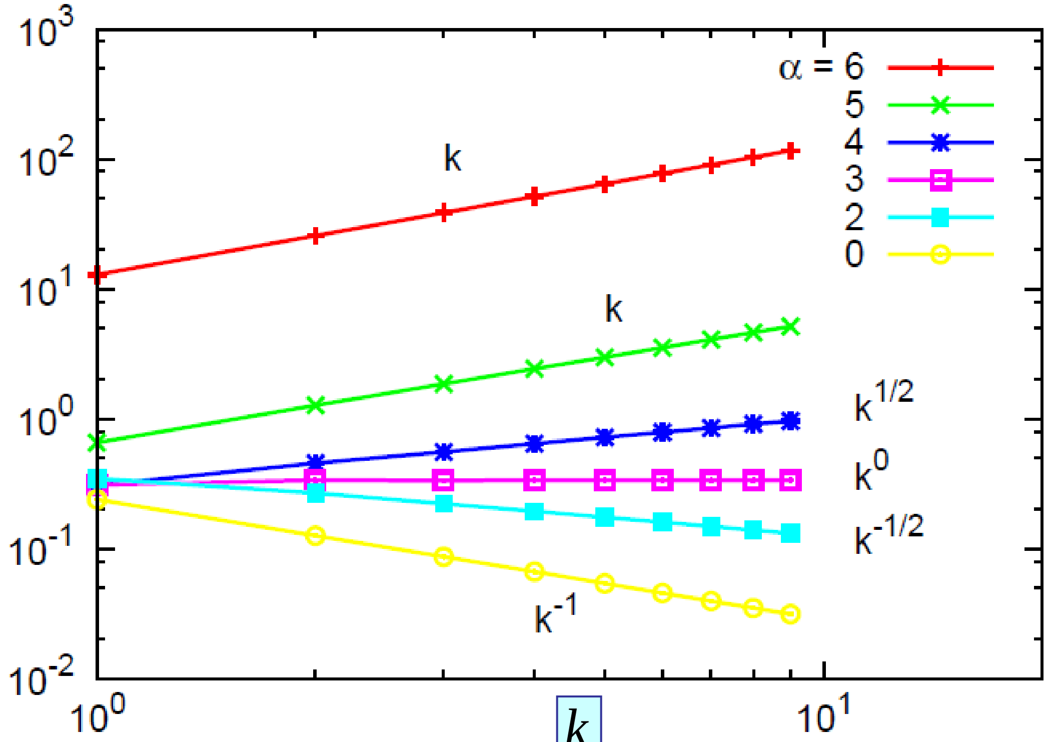
# Gap (unscaled)



# Gap between the k-th and k+1st largest

$d_k = x_k - x_{k+1}$



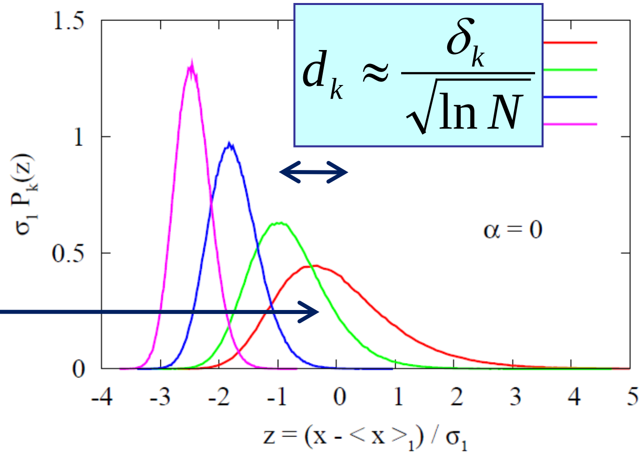

$N = 16384$

$k$

$0 \le \alpha < 1$ | $1 < \alpha < 5$ | $5 < \alpha$

$d_k \sim k^{-1}$ | $d_k \sim k^{(\alpha-3)/2}$ | $d_k \sim k$

# Phenomenology for $d_k = x_k - x_{k+1}$




$$d_k \approx \frac{\delta_k}{\sqrt{\ln N}}$$

$$b_N \sim \sqrt{\ln N}$$

$$0 \le \alpha < 1$$

$$\sqrt{\ln N} \approx b_N = \sum_{k=1}^{N} d_k = \sum_{k=1}^{N} \frac{\delta_k}{\sqrt{\ln N}}$$

assumption

$$\sum_{k=1}^{N} \delta_k \approx \ln N \rightarrow \delta_k \sim k^{-1}$$

$$1 < \alpha < 5$$

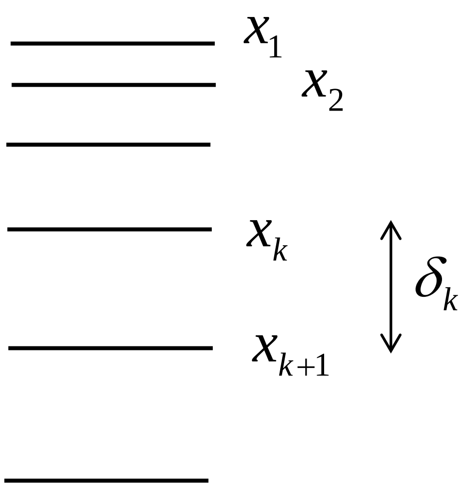
$$b_N \sim N^{(\alpha-1)/2} \qquad d_k \approx \text{const} \cdot \delta_k \qquad \sum_{k=1}^{N} \delta_k \approx N^{(\alpha-1)/2} \rightarrow \delta_k \sim k^{(\alpha-3)/2}$$

$$5 < \alpha$$

$$b_N \sim N^{(\alpha-1)/2} \qquad d_k \approx N^{(\alpha-5)/2} \delta_k \qquad \sum_{k=1}^{N} \delta_k \approx N^2 \rightarrow \delta_k \sim k$$

$$N^{(\alpha-5)/2} \sum_{k=1}^{N} \delta_k \approx N^{(\alpha-1)/2}$$

## Spectrum: $\varepsilon_k = x_1 - x_k$ scaled appropriately

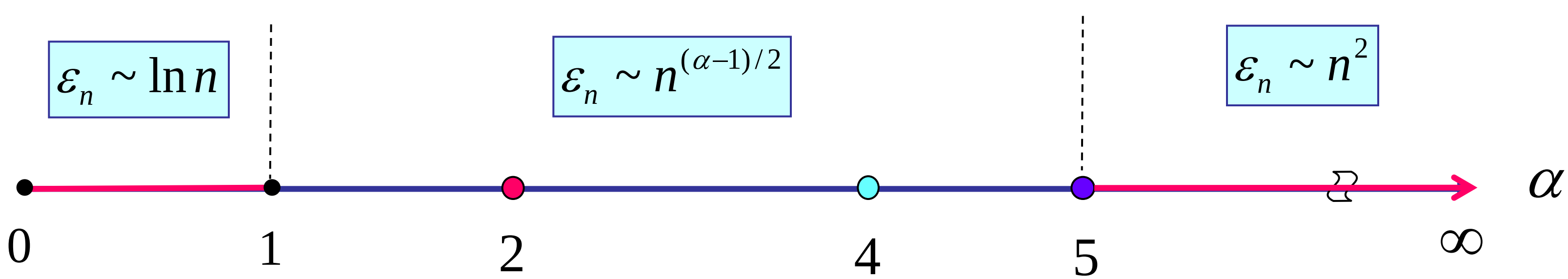



$$\varepsilon_n = \sum_{k=1}^{n} \tilde{d}_k = \sum_{k=1}^{n} \delta_k$$

$0 \le \alpha < 1$ $\quad \sum_{k=1}^{N} \delta_k \approx \ln N \quad \rightarrow \quad \varepsilon_n \sim \ln n$

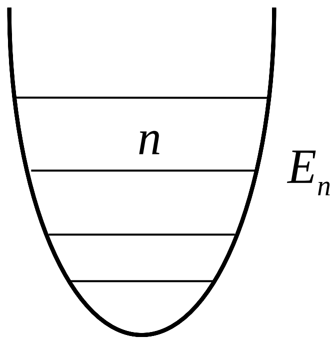
$1 < \alpha < 5$ $\quad \sum_{k=1}^{N} \delta_k \approx N^{(\alpha-1)/2} \quad \rightarrow \quad \varepsilon_n \sim n^{(\alpha-1)/2}$

$5 < \alpha$ $\quad \sum_{k=1}^{N} \delta_k \approx N^2 \quad \rightarrow \quad \varepsilon_n \sim n^2$



# Comparing with QM spectra

$U(x) = g\,|x|^{\kappa}$

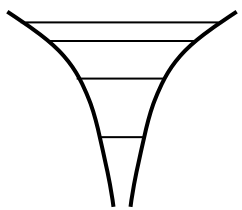



Dimensional analysis: $E_n(m,g,h) \sim h^{\frac{2\kappa}{2+\kappa}}$

Quasi-classical limit: $\oint p\,dq = nh$

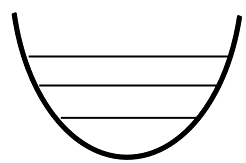
$E_n \sim n^{\frac{2\kappa}{2+\kappa}}$

$\sim \ln|x|$

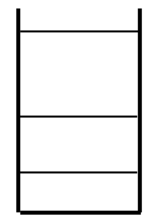


$0 \le \alpha < 1 \qquad \kappa = 0$

$U(x) \sim x^2$



$\alpha = 3 \qquad \kappa = 2$



$5 < \alpha \qquad \kappa = \infty$

$\varepsilon_n \sim \ln n$

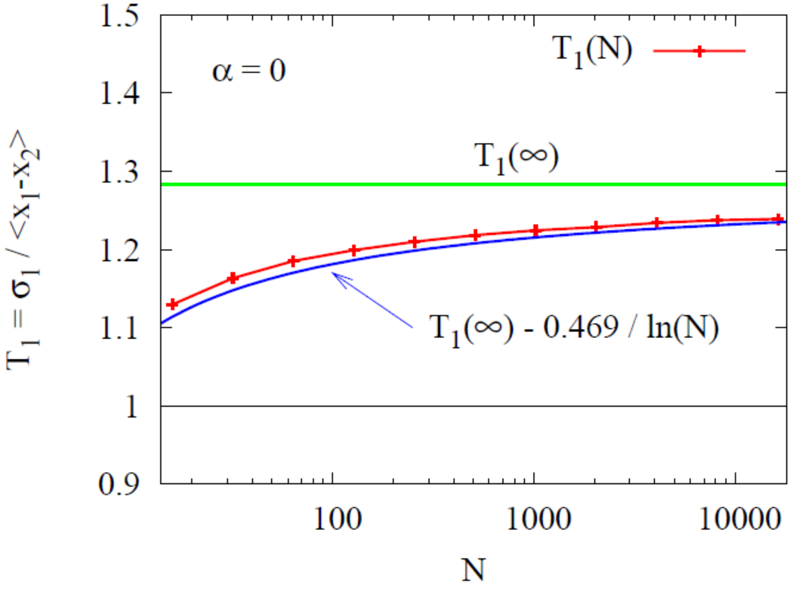
$\varepsilon_n \sim n^{(\alpha-1)/2}$

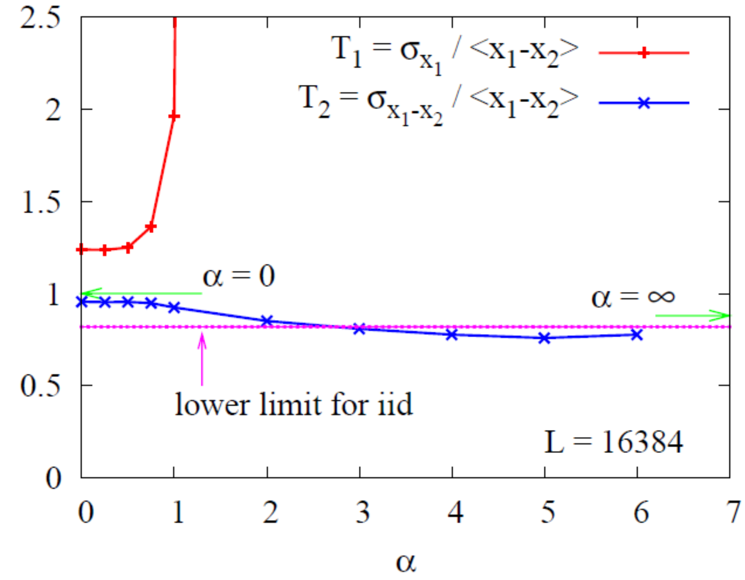
$\varepsilon_n \sim n^2$

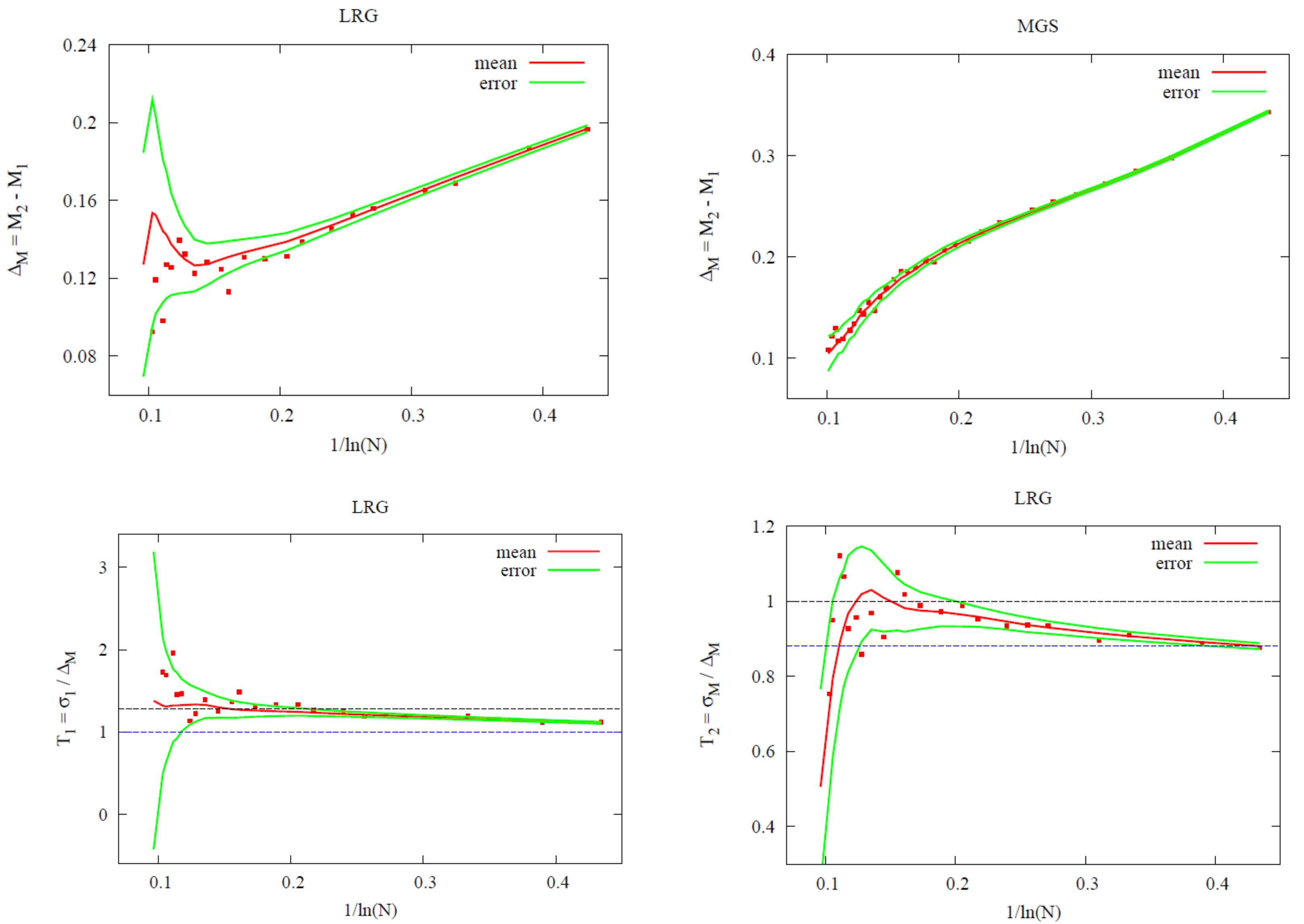
# Tremain-Richstone ratios

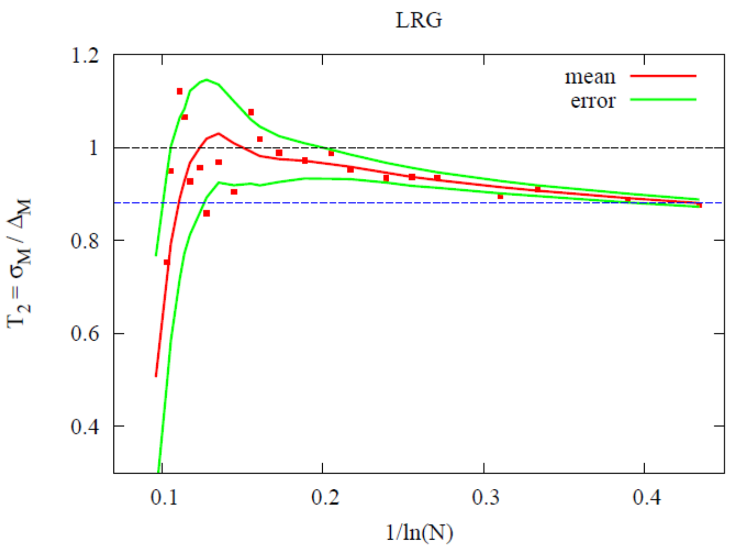
$$T_1 = \frac{\sqrt{\langle x_1^2 \rangle - \langle x_1 \rangle^2}}{\langle x_1 - x_2 \rangle} \geq 1$$

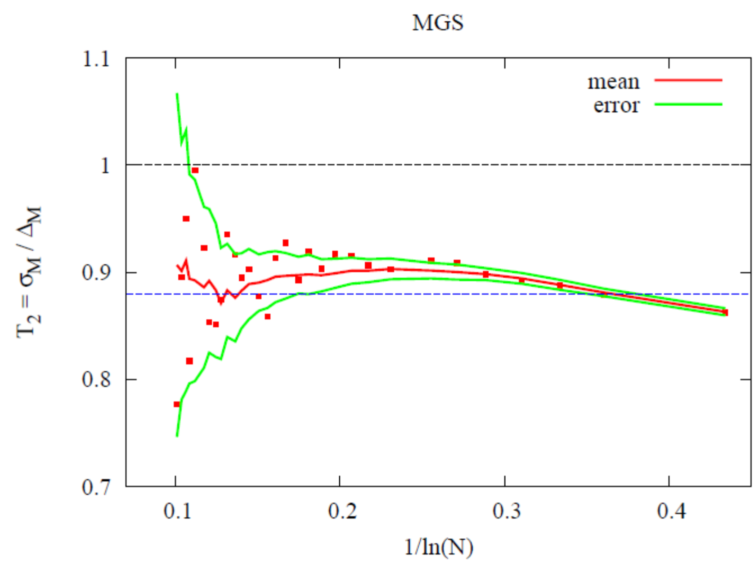
$$T_2 = \frac{\sqrt{\langle (x_1 - x_2)^2 \rangle - \langle x_1 - x_2 \rangle^2}}{\langle x_1 - x_2 \rangle} \geq 0.82$$




LRG
mean
error
0.24
0.2
0.16
0.12
0.08
$\Delta_M = M_2 - M_1$
0.1
0.2
0.3
0.4
1/ln(N)
MGS
mean
error
0.4
0.3
0.2
0.1
$\Delta_M = M_2 - M_1$
0.1
0.2
0.3
0.4
1/ln(N)
LRG
mean
error
3
2
1
0
$T_1 = \sigma_1 / \Delta_M$
0.1
0.2
0.3
0.4
1/ln(N)
LRG
mean
error
1.2
1
0.8
0.6
0.4
$T_2 = \sigma_M / \Delta_M$
0.1
0.2
0.3
0.4
1/ln(N)


LRG
mean
error
1.2
1
0.8
0.6
0.4
0.1
0.2
0.3
0.4
$T_2 = \sigma_M / \Delta_M$
1/ln(N)


MGS
mean
error
1.1
1
0.9
0.8
0.7
0.1
0.2
0.3
0.4
$T_2 = \sigma_M / \Delta_M$
1/ln(N)