Patterns and Fronts

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Introduction

(1) Why is there something instead of nothing?

Homogeneous vs. inhomogeneous systems
Deterministic vs. probabilistic description
Instabilities and symmetry breakings in homogeneous systems

(2) Can we hope to describe the myriads of patterns?

Notion of universality near a critical instability.
Common features of emerging patterns.
  Example: Benard instability and visual hallucinations.
Notion of effective long-range interactions far from equilibrium.
Scale-invariant structures.

(3) Should we use macroscopic or microscopic equations?

Relevant and irrelevant fields -- effects of noise.
  Arguments for the macroscopic.
  Example: Snowflakes and their growth.
Remanence of the microscopic: Anisotropy and singular perturbations.

Patterns from stability analysis

(1) Local and global approaches.

Problem of relative stability in far from equilibrium systems.

(2) Linear stability analysis.

Stationary (fixed) points of differential equations.
Behavior of solutions near fixed points: stability matrix and eigenvalues.
  Example: Two dimensional phase space structures
    Lotka-Volterra equations, story of tuberculosis
Breaking of time-translational symmetry: hard-mode instabilities
  Example: Hopf bifurcation: Van der Pole oscillator
Soft-mode instabilities: Emergence of spatial structures
  Example: Chemical reactions - Brusselator.

(3) Critical slowing down and amplitude equations for the slow modes.

Landau-Ginzburg equation with real coefficients.
Symmetry considerations and linear combination of slow modes.
Boundary conditions - pattern selection by ramp.

(4) Weakly nonlinear analysis of the dynamics of patterns.

Secondary instabilities of spatial structures.
Eckhaus and zig-zag instability, time dependent structures.

(5) Complex Landau-Ginzburg equation

Convective and absolute instabilities of patterns.
Benjamin-Feir instability - spatio-temporal chaos.
One-dimensional coherent structures, noise sustained structures.

Patterns from moving fronts

(1) Importance of moving fronts: Patterns are manufactured in them.

Examples: Crystal growth, DLA, reaction fronts.
Dynamics of interfaces separating phases of different stability.
Classification of fronts: pushed and pulled.

(2) Invasion of an unstable state.

- Velocity selection.
  - Example: Population dynamics.
- Stationary point analysis of the Fisher-Kolmogorov equation.
- Wavelength selection.
  - Example: Cahn-Hilliard equation and coarsening waves.

(3) Diffusive fronts.

Liesegang phenomena (precipitation patterns in the wake of diffusive reaction fronts - a problem of distinguishing the general and particular).

Literature

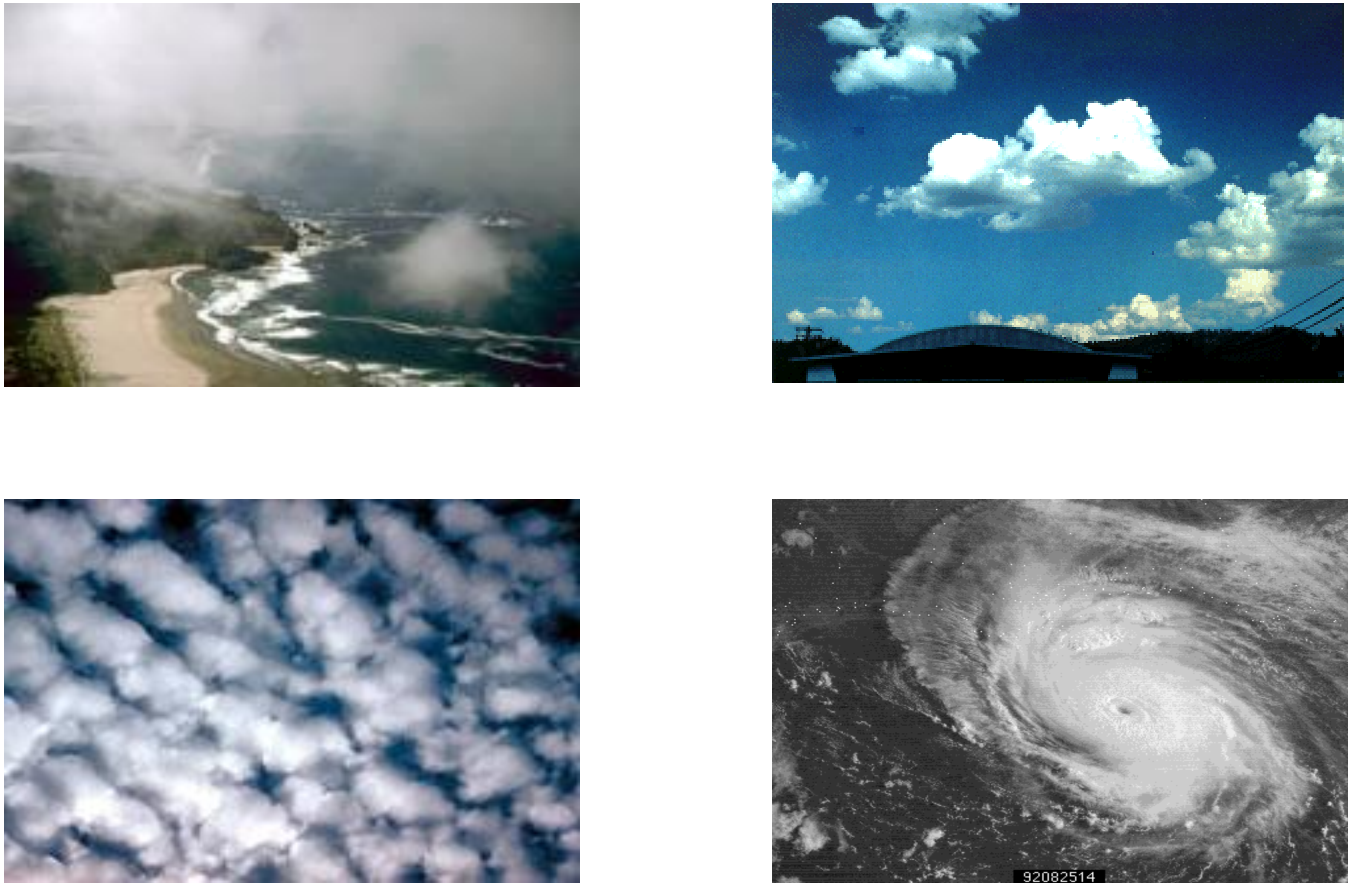
M. C. Cross and P. C. Hohenberg, **Pattern Formation Outside of Equilibrium**,
Rev. Mod. Phys. **65**, 851 (1993).

J. D. Murray, **Mathematical Biology**, (Springer, 1993; ISBN-0387-57204).

W. van Saarloos, **Front propagation into unstable state,**
Physics Reports, **386** 29-222 (2003)

Why is there Something instead of Nothing? (Leibniz)

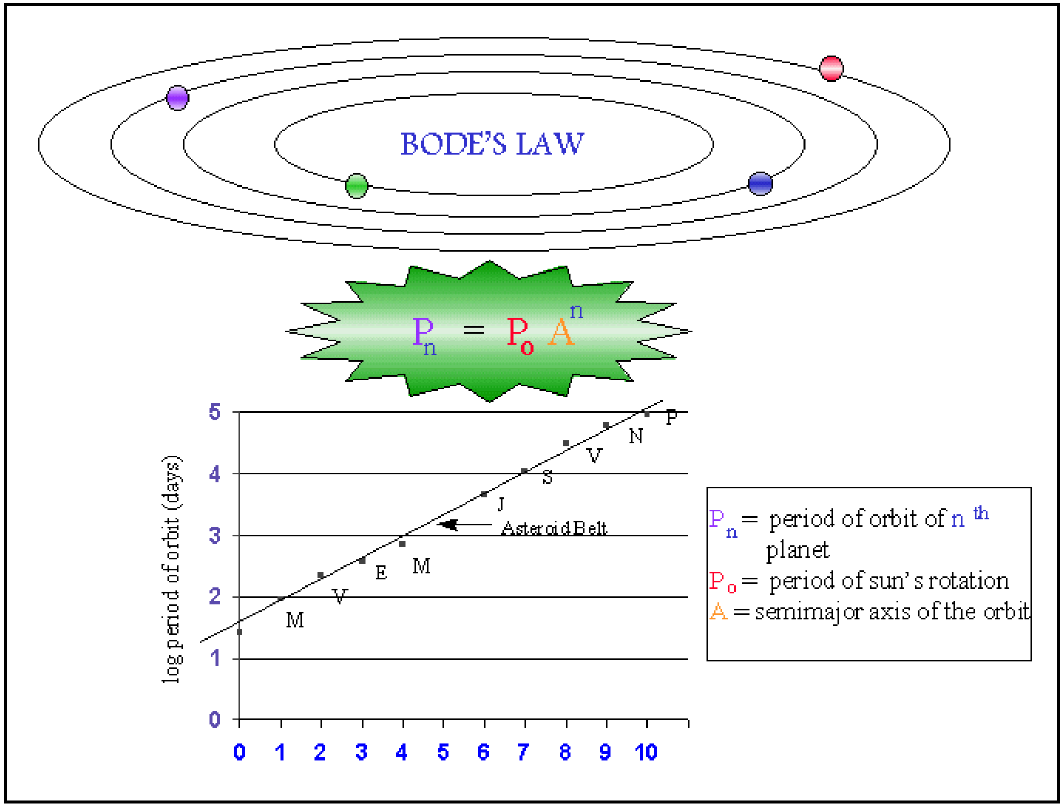
Homogeneous (amorphous) vs. inhomogeneous (structured)



Actors
and
spectators
(N. Bohr)

Deterministic vs. probabilistic aspects I.

The question of the origins of order:



(Cornell Universty)

Bishop to Newton:

Now that you discovered the laws governing the motion of the planets, can you also explain the regularity of their distances from the Sun?

Newton to Bishop:

I have nothing to do with this problem. The initial conditions were set by God.

?

Titius-Bode law

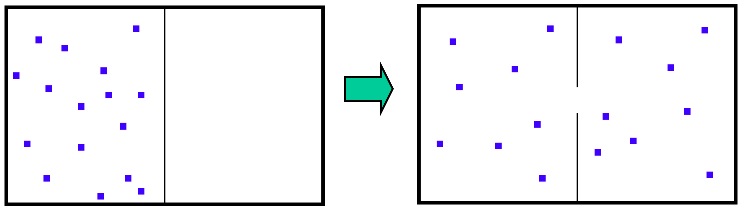
Deterministic vs. probabilistic aspects II.

The question of the origins of order:

Mechanics, electrodynamics, quantum mechanics: initial conditions

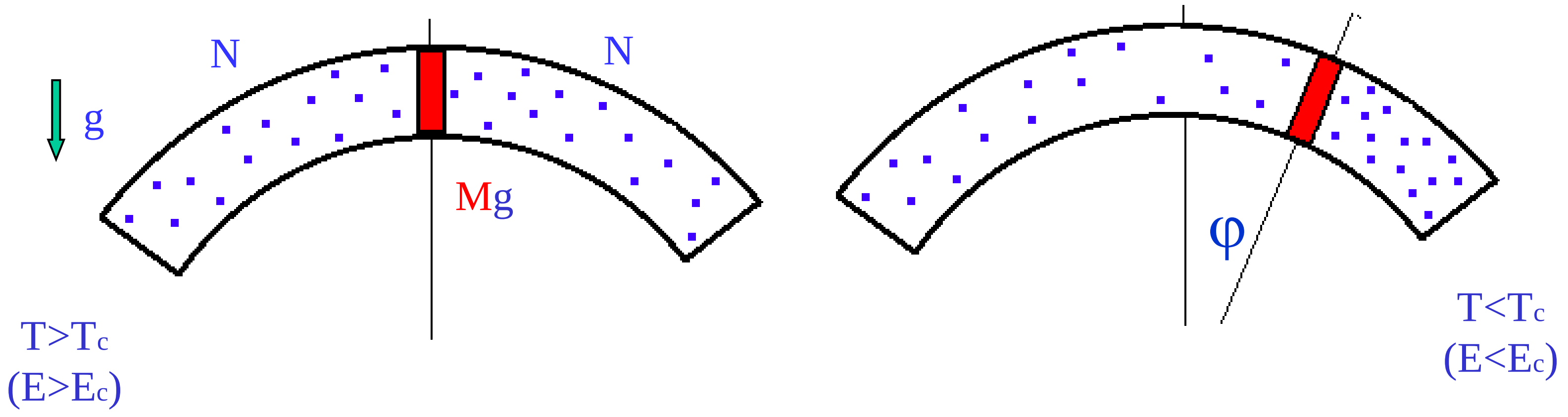
Thermodynamics, statistical mechanics: S=max
(equilibrium is independent of initial conditions) (at given constraints)

Stability



Disorder wins?

Order in equilirium

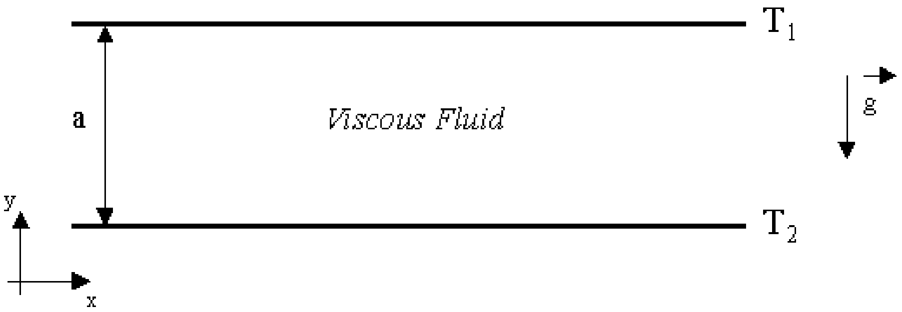


Instability - symmetry breaking - critical slowing down

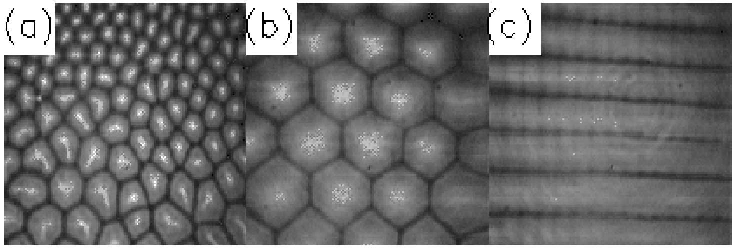
Instabilities and Symmetry Breakings

Basic approach: Understand more complex through studies of (symmetry breaking) instabilities of less complex

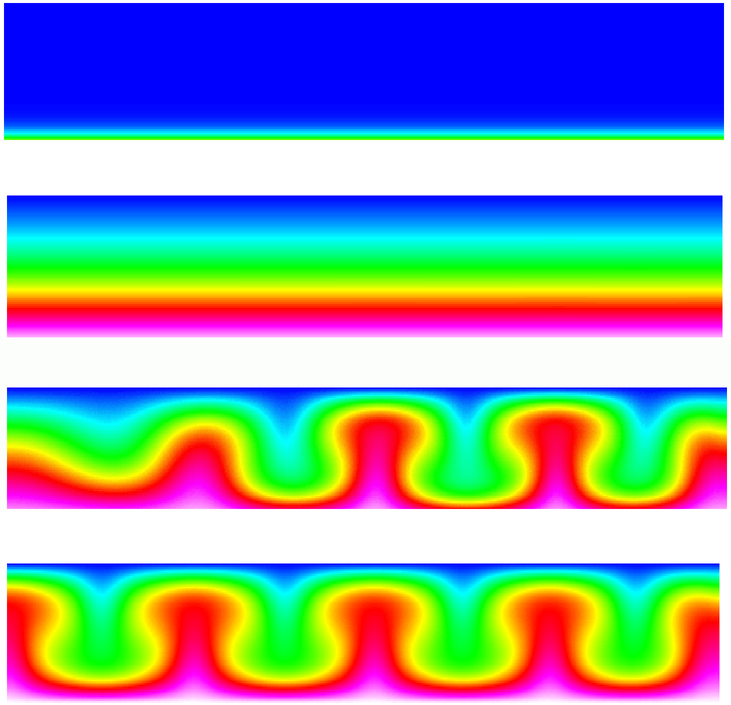
Rayleigh-Bénard:



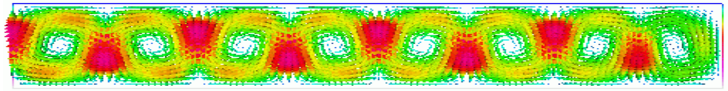
M.Schatz (shadowgraph images of convection patterns):



temperature field



velocity field



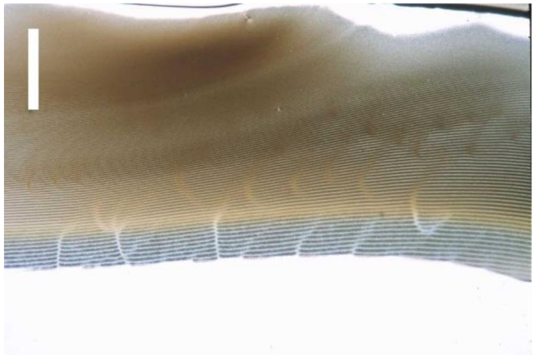
(Elmer Co.)

The wonderful world of stripes

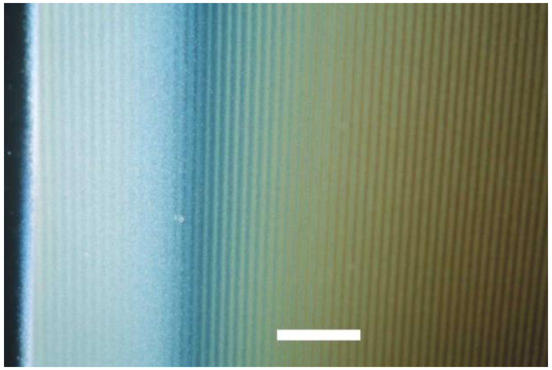
Clouds

Characteristic length:

$\sim 10^{2}$ m



P. Hantz



Precipitation patterns in gels

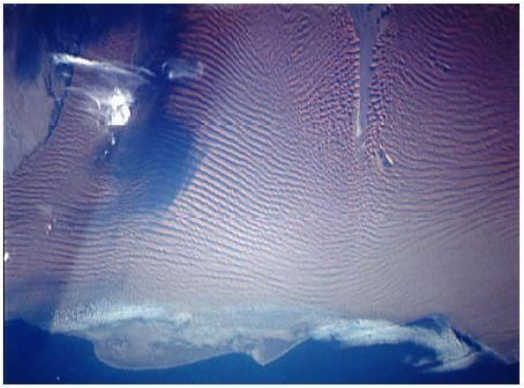
$CuCl_2 + NaOH \longrightarrow$

$\underline{CuO}$ + ...

$\sim 10^{-4}$m

The massive white dunes of Sand Mountain, southeast of Fallon, Nevada. This is one of the few "booming dunes"

NASA



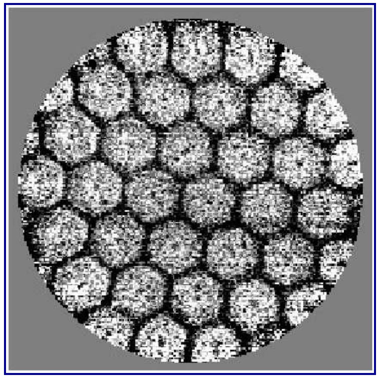
Sand dunes

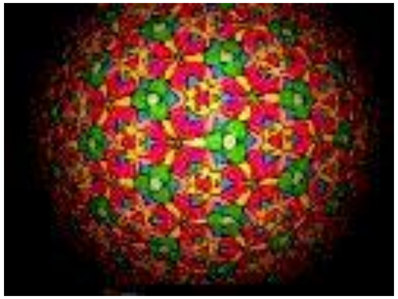
$\sim 10^{-1} - 10^{4}$m

Visual Hallucinations and the Bénard Instability

Bénard experiments (G. Ahlers et al.)

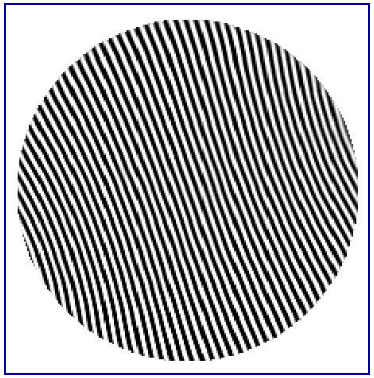
Visual hallucinations (H. Kluver)



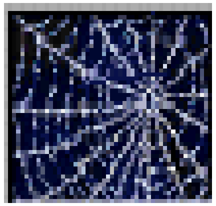


Caleidoscope

(lattice, network, grating honeycomb)

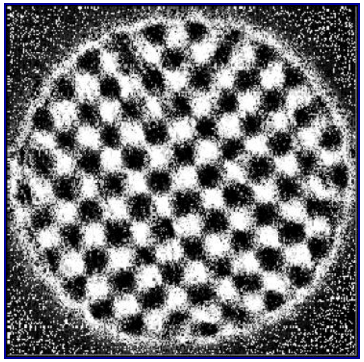


tunnel

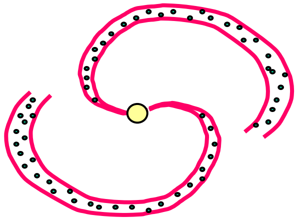


funnel

spiral

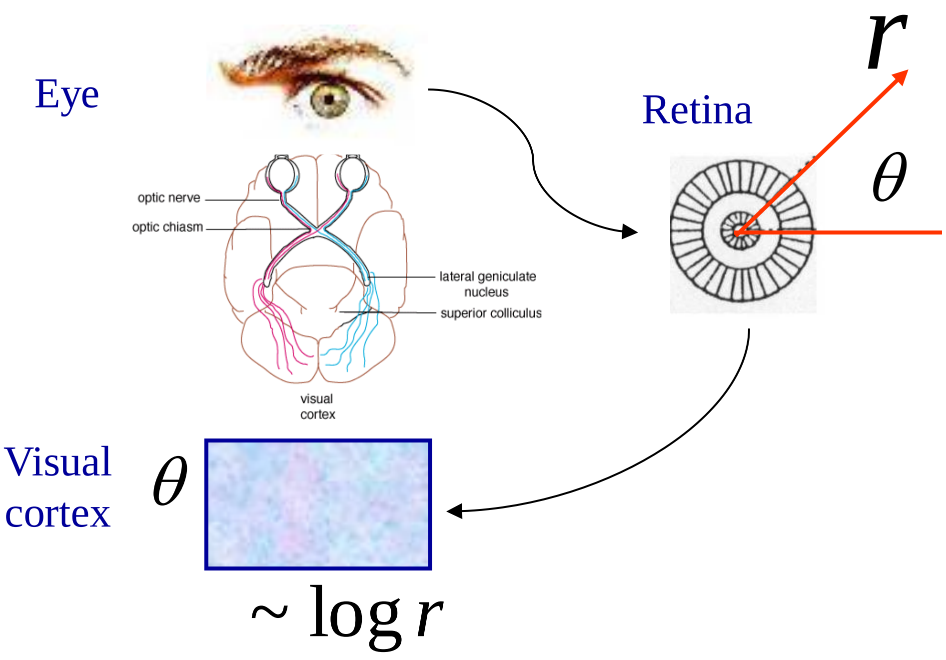


cobweb



Visual hallucinations: retina ⟷ visual cortex mapping

J. D. Cowan



Eye

Retina

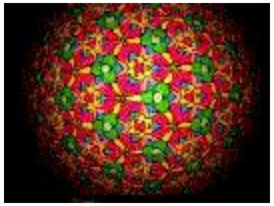
r

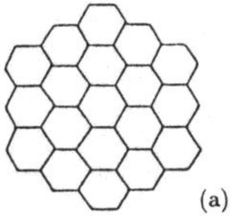
θ

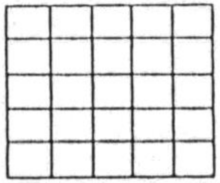
Visual cortex

θ

$\sim \log r$

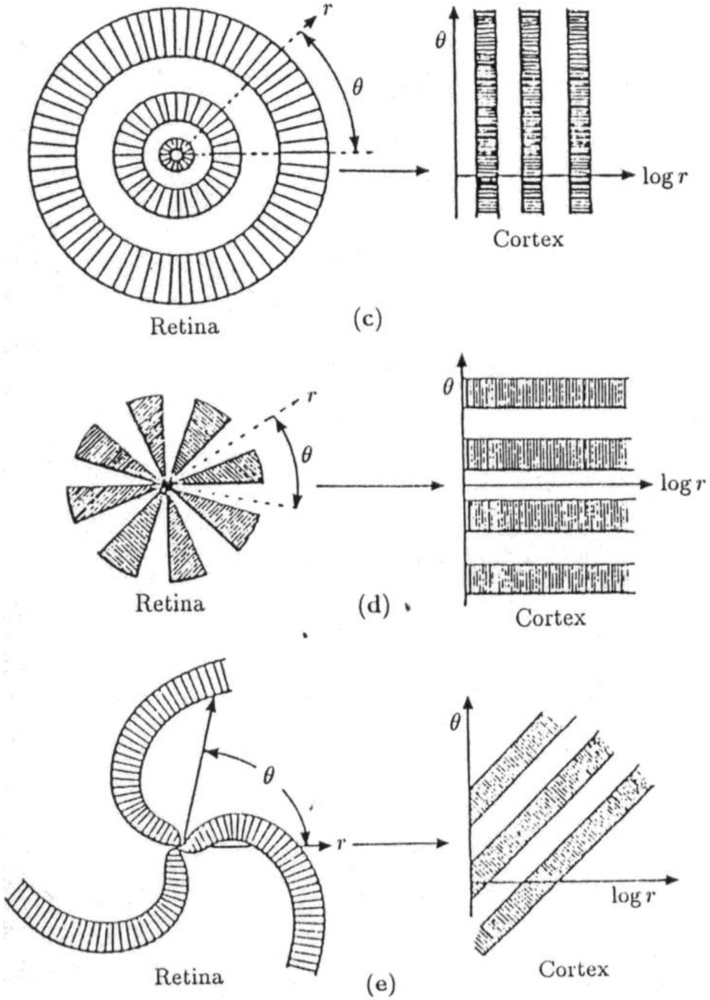






Retina

Visual cortex

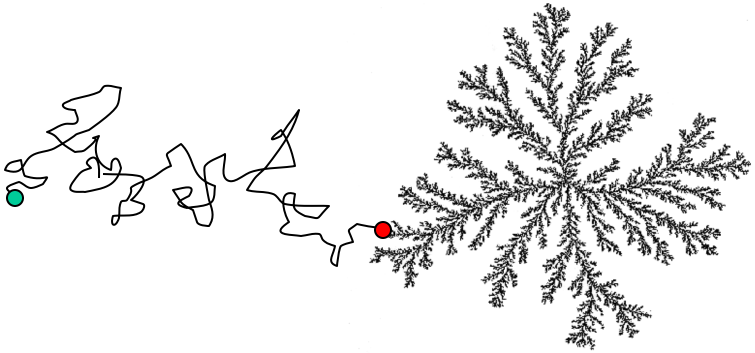


Scale Invariant Structures

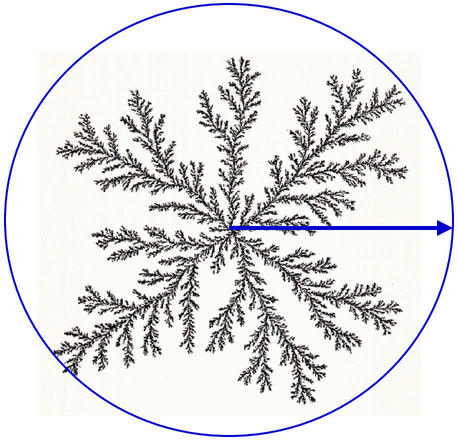
Oak tree

MgO2 in Limestone **C.-H. Lam**

DLA (diffusion limited aggregation)



1 million particles



N=100 million (H. Kaufman)

$$R \approx N^{1/D}$$

Level of description: Microscopic or macroscopic?

(1) No two snowflakes are alike

(2) All six branches are alike

⇒ Parameters determining growth fluctuate on lengthscales larger than 1*mm*.

(3) Sixfold symmetry

⇒ Microscopic structure is relevant on macroscopic scale.

(4) Twelvefold symmetry (not very often)

⇒ Initial conditions may be remembered.

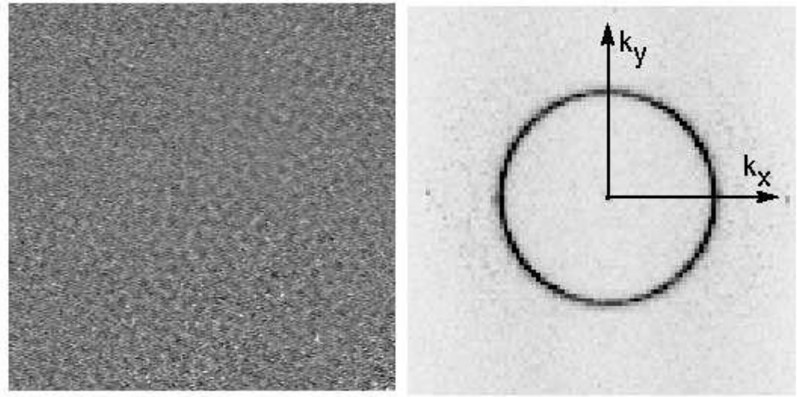
Fluctuations and Noise

disorder
homogeneous

instability
large fluctuations

order

Rayleigh-Benard near but below the convection instability:



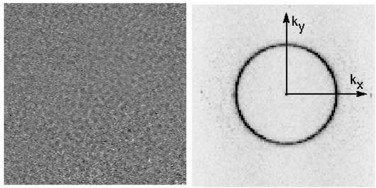
G. Ahlers et al.

Power spectrum

$$S(k) \sim \langle | \rho_{\mathrm{k}} |^2 \rangle$$

$$\rho_k = \frac{1}{V} \sum_x e^{ikx} \rho(x)$$

Emergence of spatial structures: Soft mode instabilities



$$n \rightarrow n(t) \rightarrow n(x,t)$$

$$\dot{n} = f(n, \partial_x n, \partial_x^2 n, \ldots, \lambda)$$

Spatial mixing: convection, diffusion, … ⟹ Reaction-diffusion systems

$$\dot{n}_1 = D_1 \Delta n_1 + f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = D_2 \Delta n_2 + f_2(n_1, n_2, \lambda)$$

Chemical reactions in gels
(model example: Brussellator)

Stability analysis:

(1) Stationary homogeneous solutions:

$$f_1(n_1^*, n_2^*, \lambda) = 0$$
$$f_2(n_1^*, n_2^*, \lambda) = 0$$

Stab

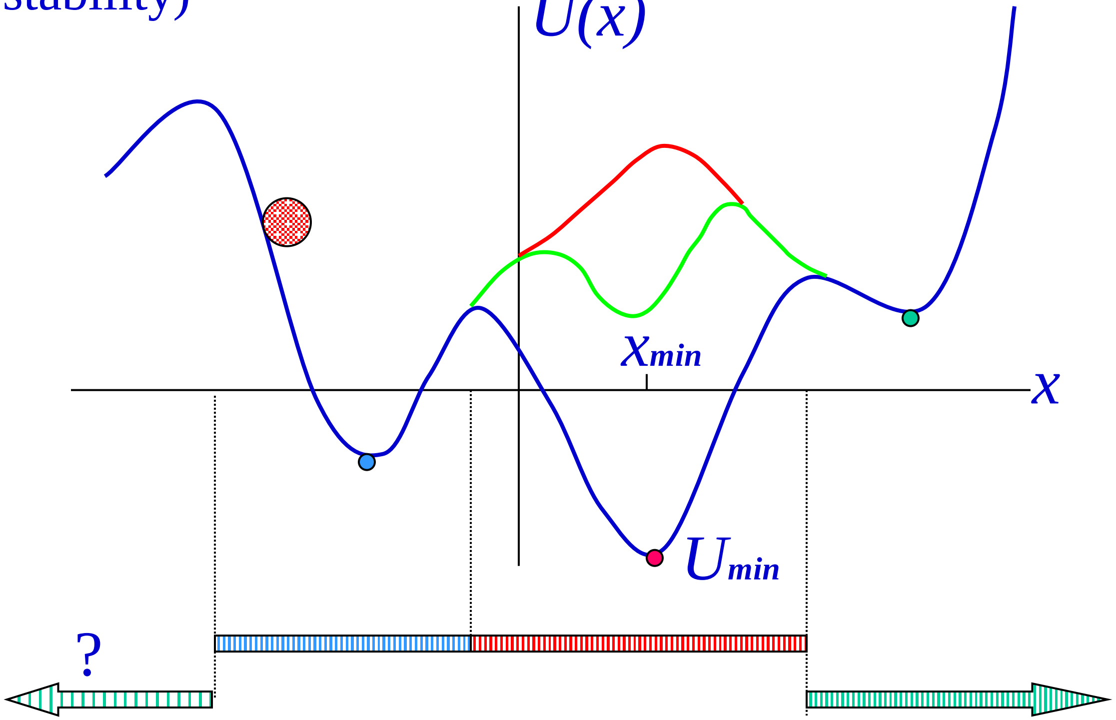
Local and global approaches

Stability (absolute and relative stability)

$$\ddot{x} = -\frac{dU}{dx}$$

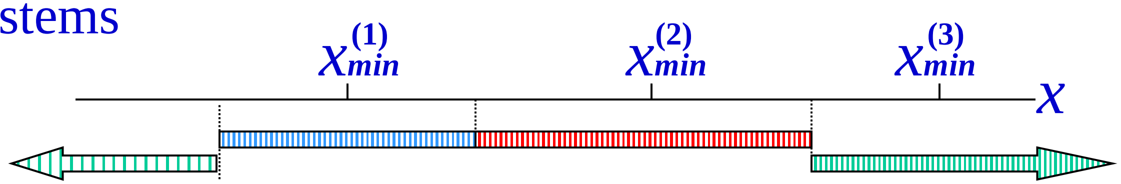
Stationary points

$$\frac{dU}{dx} = 0$$



Regions of attraction of stationary points

The problem of non-potential systems



Linear stability analysis

$$\dot{n}_1 = f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = f_2(n_1, n_2, \lambda)$$

Assumption: The system is desribed by autonomous differential equations

fields control parameter

Stationary (fixed) points of the equations:

$$n_1^*, n_2^*$$

$$f_1(n_1^*, n_2^*, \lambda) = 0$$
$$f_2(n_1^*, n_2^*, \lambda) = 0$$

Behavior of solutions near fixed points

Linearization

$$\begin{aligned} \dot n_1 &= f_1(n_1,n_2,\lambda) \\ \dot n_2 &= f_2(n_1,n_2,\lambda) \end{aligned}$$

n_1^*, n_2^*

$$\begin{aligned} n_1 &= n_1^* + \delta n_1(t) \\ n_2 &= n_2^* + \delta n_2(t) \end{aligned}$$

$$\begin{pmatrix} \delta\dot n_1 \\ \delta\dot n_2 \end{pmatrix} = \mathbf{A}_\lambda \begin{pmatrix} \delta n_1 \\ \delta n_2 \end{pmatrix}$$

Stability matrix

Diagonalization

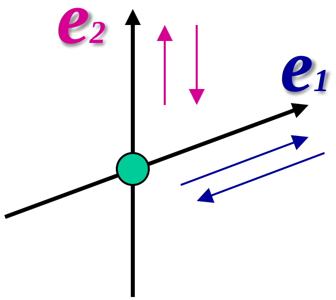
$$\tilde{\mathbf{A}}_\lambda = \begin{pmatrix} \omega_{1\lambda} & 0 \\ 0 & \omega_{2\lambda} \end{pmatrix}$$

Eigenvalues

Solution

$$\begin{pmatrix} \delta n_1 \\ \delta n_2 \end{pmatrix} = c_1 \mathbf{e_1}\, e^{\omega_{1\lambda} t} + c_2 \mathbf{e}_2\, e^{\omega_{2\lambda} t}$$

Eigenvectors

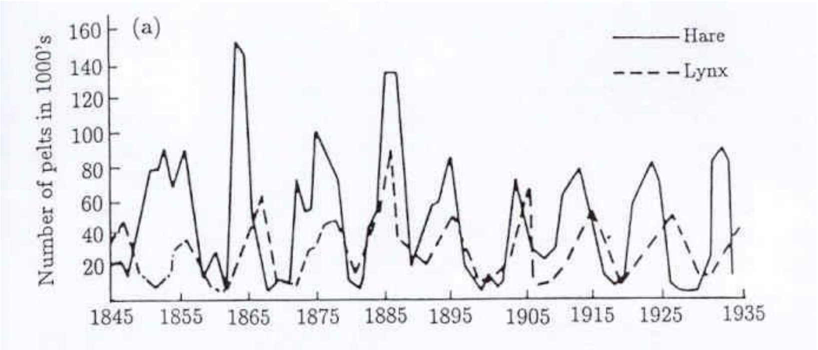


Lotka-Volterra systems: Hare-lynx problem

Lotka 1925 (osc. chem. react.)
Volterra 1926 (fish pop.)

n_1 - concentration of hare (rabbits)

n_2 - conc. of lynx (foxes)



αn_1 - rabbits eat grass and multiply

($\alpha = 1$ sets the time-scale)

$-\lambda n_2$ - foxes perish without eating

$-\beta n_1 n_2$ - rabbits perish when meeting foxes ($\beta = 1$ sets the n_1 -scale)

$\gamma n_1 n_2$ - foxes multiply when meeting rabbits ($\gamma = 1$ sets the n_2 -scale)

$$\dot{n}_1 = n_1 - n_1 n_2$$
$$\dot{n}_2 = n_1 n_2 - \lambda n_2$$

$-\kappa n_1^2$ - finite grass supply

Fixed points:

$$n_1^* = 0, \; n_2^* = 0$$

$$n_1^* = \lambda, \; n_2^* = 1$$

Hare-lynx problem: Fixed point structure

$$\dot{n}_1 = n_1 - n_1 n_2$$
$$\dot{n}_2 = n_1 n_2 - \lambda n_2$$

$$n_1^* = 0,\ \ n_2^* = 0$$

Doom

$$n_1^* = \lambda,\ \ n_2^* = 1$$

Coexistence

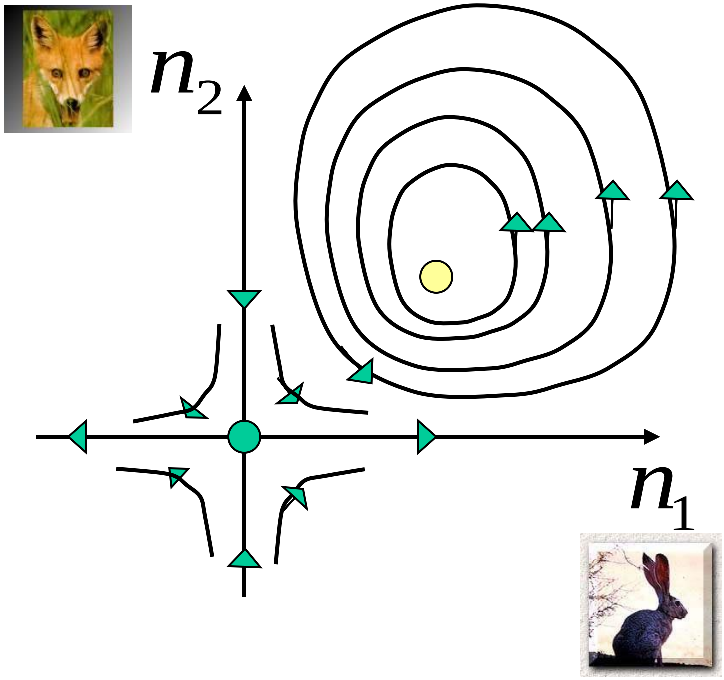
$$\mathbf{A}_\lambda \Rightarrow$$

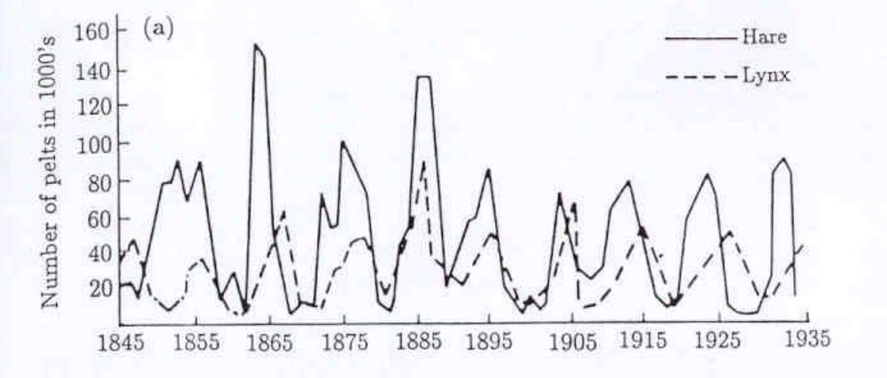
$$\begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix}$$

$$\omega_1 = 1 \quad \omega_2 = -\lambda$$

$$\omega_{1,2} = \pm i\sqrt{\lambda}$$





Problems of fluctuations and discreteness

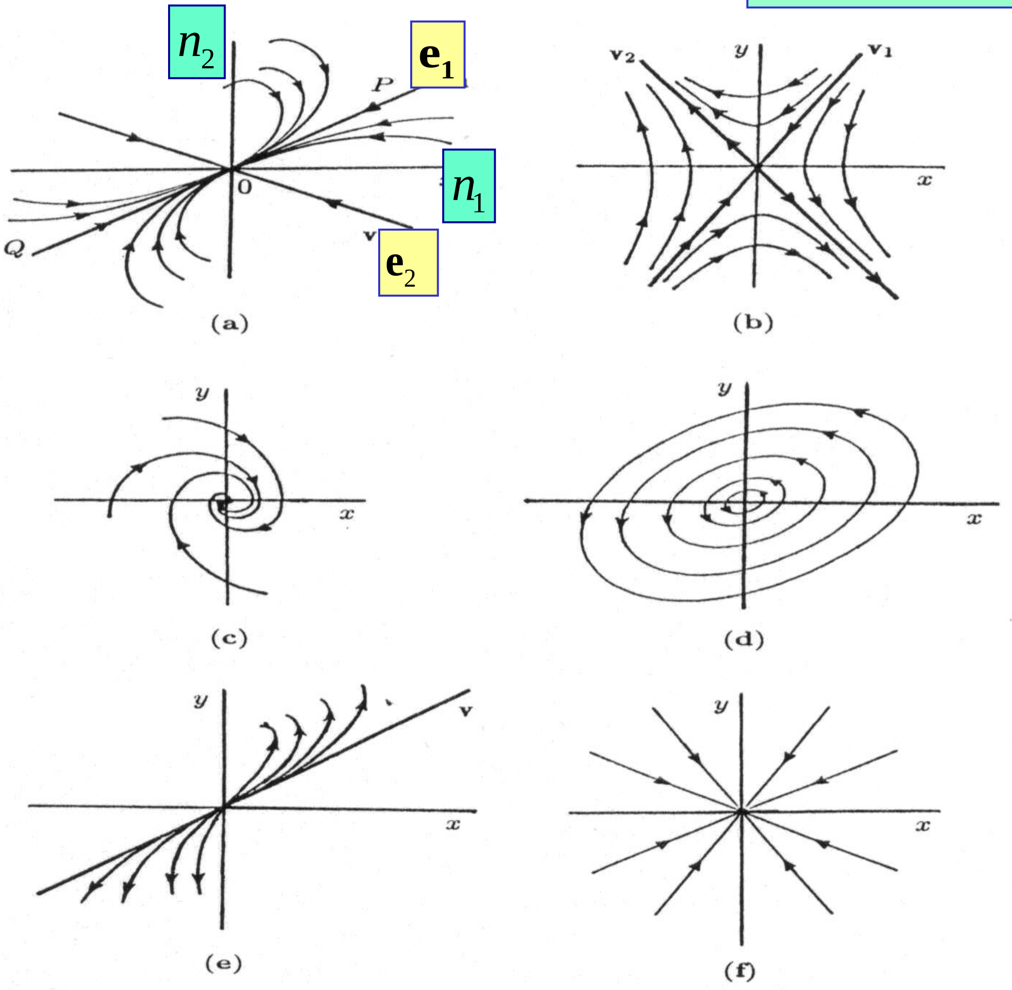
Fixed point structures

$$\dot{n}_1 = f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = f_2(n_1, n_2, \lambda)$$

$$\omega_1, \omega_2, \mathbf{e}_1, \mathbf{e}_2$$

Re Im

$$\omega \rightarrow +,0,- \quad \pm i$$



$\text{Re} \ - < -$

Node I

$\text{Re} \ + \ -$

Saddle

$\text{Re} = - \quad \text{Im}$

Spiral

$\text{Re} = 0 \quad \text{Im}$

Centre

$\text{Re} + = +$

Node II

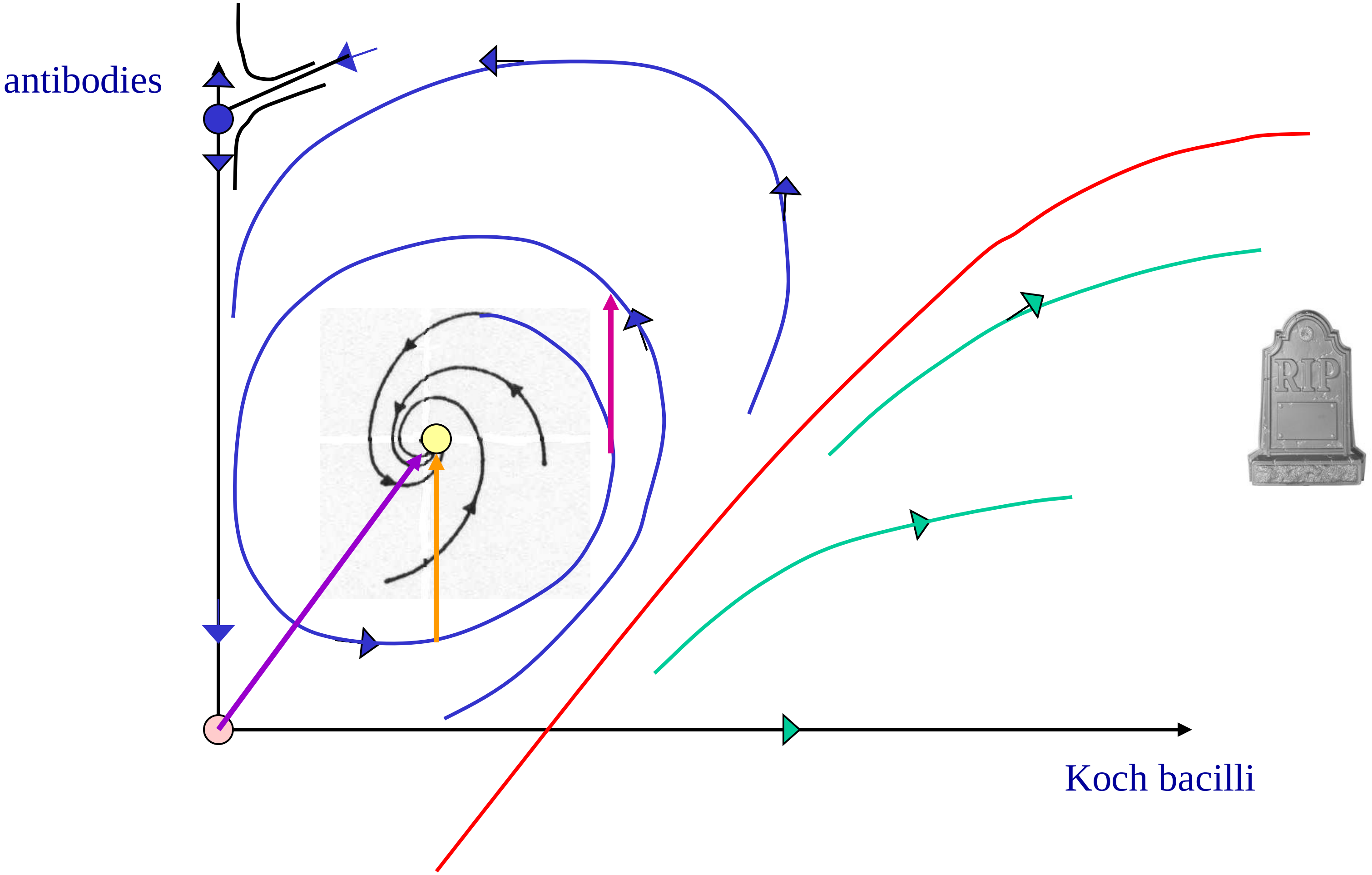
$\text{Re} \ - = -$

Star

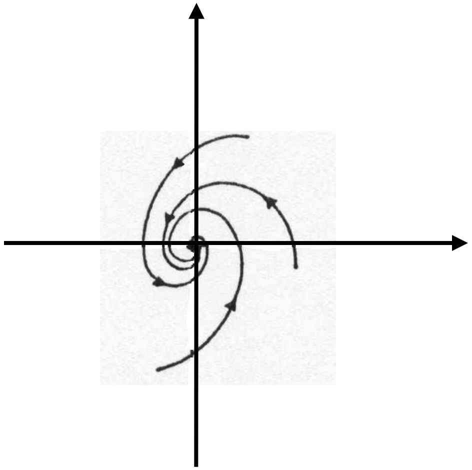
Fixed point structures and the problem of tuberculosis

Cause: Koch bacillus; Treatment: antibiotics; Immunity by vaccination

Characteristics: periodicity in the course of illness



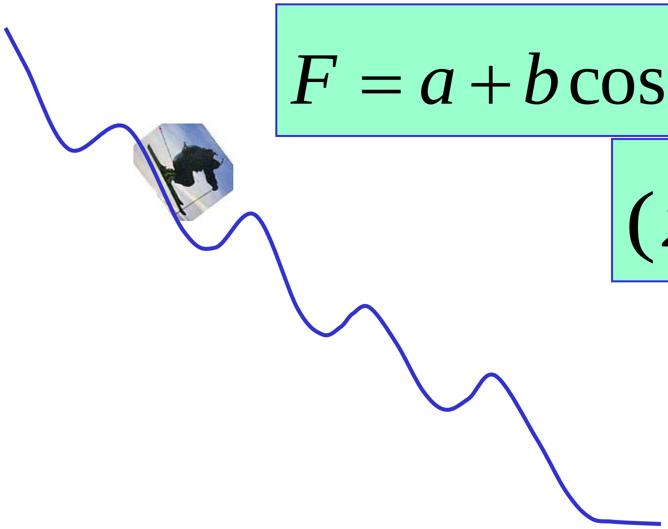
Breaking of time-translational symmetry: Limit cycles

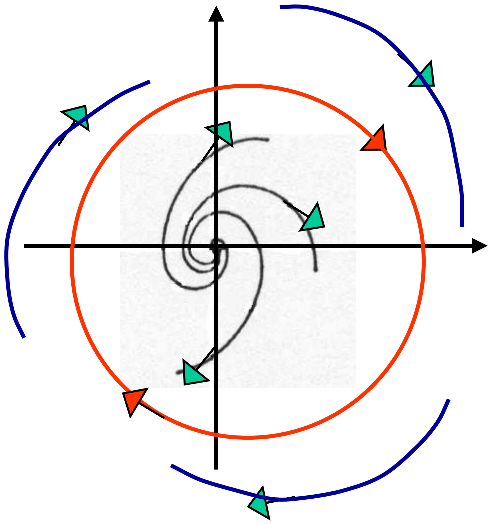


Example: Skier on a wavy slope

$$F = a + b\cos(x) - c\,\dot{x}$$

$$(x, \dot{x})$$



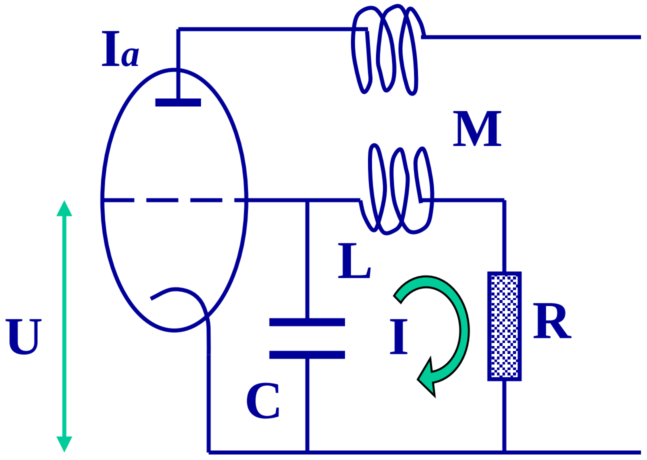


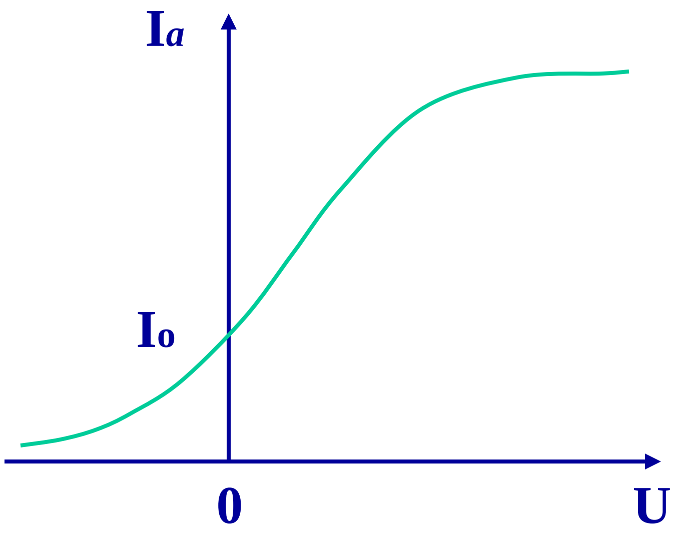
Example: Van der Pole oscillator ▶

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x$$

$$(\dot{x} \sim I, x \sim U)$$

Van der Pole oscillator





$$I_a = I_0 + sU - \frac{g}{3}U^3$$

$$L\frac{dI}{dt} + M\frac{dI_a}{dt} + IR + \frac{1}{C}\int_0^t I d\tau = 0$$

$$x = \sqrt{\frac{Mg}{Ms - RC}}U$$

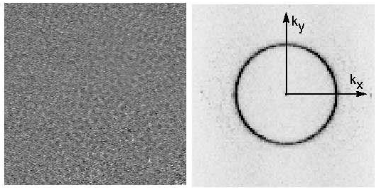
$$I = C\sqrt{\mu}\dot{x}$$

$$t \to \sqrt{LC}t'$$

$$\mu = \frac{Ms - RC}{\sqrt{LC}}$$

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x$$

Emergence of spatial structures: Soft mode instabilities



$$n \rightarrow n(t) \rightarrow n(x,t)$$

$$\dot{n} = f(n, \partial_x n, \partial_x^2 n, \ldots, \lambda)$$

Spatial mixing: convection, diffusion, … ⟹ Reaction-diffusion systems

$$\dot{n}_1 = D_1 \Delta n_1 + f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = D_2 \Delta n_2 + f_2(n_1, n_2, \lambda)$$

Chemical reactions in gels
(model example: Brussellator)

▶

Stability analysis:

(1) Stationary homogeneous solutions:

$$f_1(n_1^*, n_2^*, \lambda) = 0$$
$$f_2(n_1^*, n_2^*, \lambda) = 0$$

Stab

Brusselator -

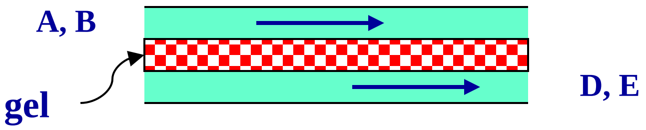
(Prigogin and Lefever, 1968)

oscillations and spatial patterns in chemical reactions

$$\mathbf{A} \xrightarrow{k_1} \mathbf{U} \qquad \mathbf{B} + \mathbf{U} \xrightarrow{k_2} \mathbf{V} + \mathbf{D}$$

$$\mathbf{2U} + \mathbf{V} \xrightarrow{k_3} \mathbf{3U} \qquad \mathbf{U} \xrightarrow{k_4} \mathbf{E}$$

Concentrations: **A, B, U(x,t), V(x,t)**



$$\dot{U} = D_u \Delta U + k_1 A - k_2 BU + k_3 U^2 V - k_4 U$$
$$\dot{V} = D_v \Delta V \qquad\qquad + k_2 BU - k_3 U^2 V$$

Brusselator - rescalings and canonical form of the equations

A ⟶ **U**

B + U ⟶ **V + D**

2U + V ⟶ **3U**

U ⟶ **E**

$$\dot{U} = D_u \Delta U + k_1 A - k_2 BU + k_3 U^2 V - k_4 U$$
$$\dot{V} = D_v \Delta V \qquad + k_2 BU - k_3 U^2 V$$

time

$$t = t' / k_4$$

space

$$x = x' \sqrt{D_u / k_4}$$

concentrations

$$U = cu \qquad V = cv \qquad c = Ak_1 / k_4$$

parameters

$$d = D_v / D_u$$

$$b = k_2 B / k_4$$

equations

$$\dot{u} = \Delta u + 1 - (1 + b)u + \lambda u^2 v$$
$$\dot{v} = d\Delta v \qquad + bu - \lambda u^2 v$$

control parameter

$$\lambda = k_3^3 k_1^2 A^2 / k_4^3$$

Brusselator II - rescalings and canonical form of the equations

A → U

B + U → V + D

2U + V → 3U

U → E

$$\dot{U} = D_u \Delta U + k_1 A - k_2 BU + k_3 U^2 V - k_4 U$$
$$\dot{V} = D_v \Delta V \qquad\qquad + k_2 BU - k_3 U^2 V$$

time

$$t = t' \tau$$

space

$$x = x' \ell$$

concentrations

$$U = cu \qquad V = c' v$$

equations

$$\frac{c}{\tau} \dot{u} = \frac{D_u c}{\ell^2} \Delta u + k_1 A - k_2 Bcu + k_3 c^2 c' u^2 v - k_4 cu$$
$$\frac{c'}{\tau} \dot{v} = \frac{D_v c'}{\ell^2} \Delta v \qquad\qquad + k_2 Bcu - k_3 c^2 c' u^2 v$$

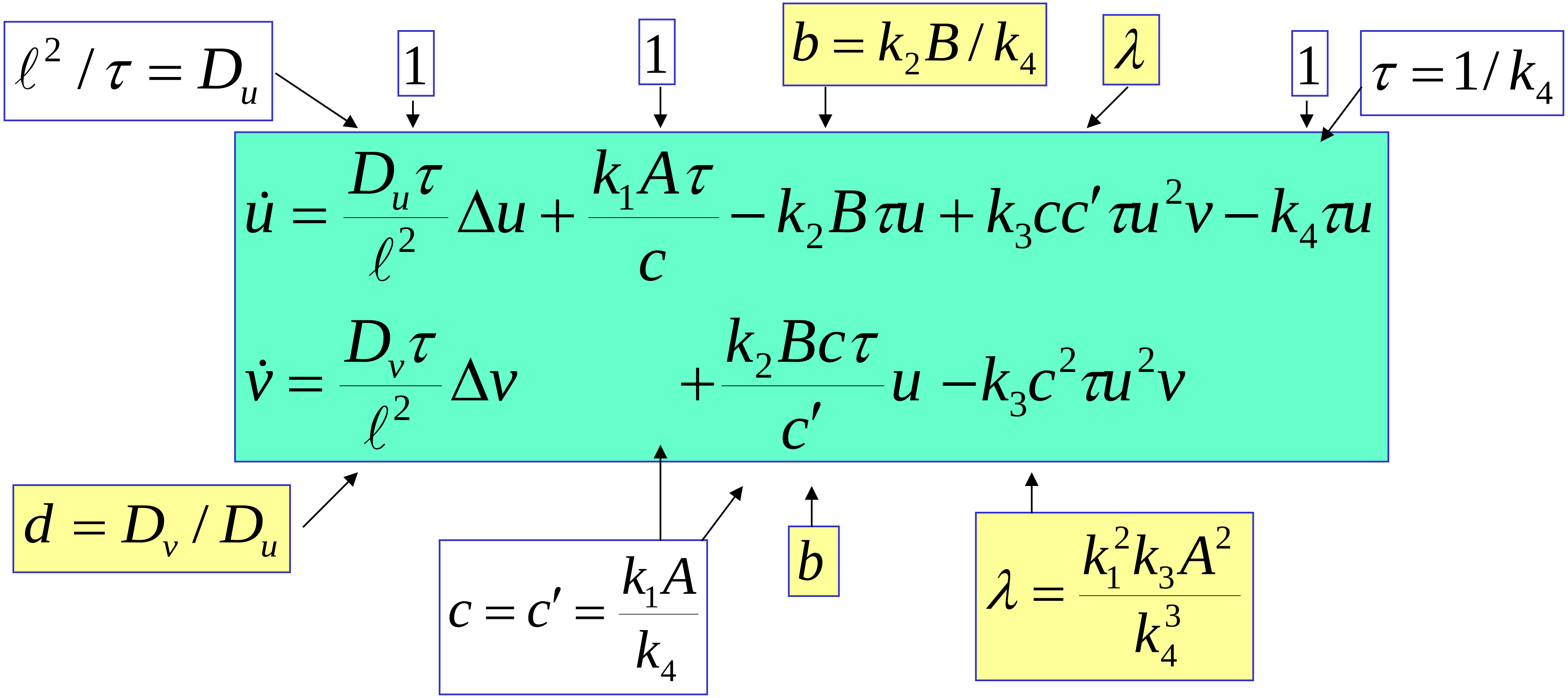
Brusselator III - rescalings and canonical form of the equations

$A \rightarrow U$ $\quad$ $B + U \rightarrow V + D$

$2U + V \rightarrow 3U$ $\quad$ $U \rightarrow E$

$$\dot{U} = D_u \Delta U + k_1 A - k_2 B U + k_3 U^2 V - k_4 U$$
$$\dot{V} = D_v \Delta V \qquad + k_2 B U - k_3 U^2 V$$

time $t = t'\tau$ space $x = x'\ell$ concentrations $U = cu \qquad V = c'v$



$\ell^2 / \tau = D_u$

$\boxed{1}$ $\quad$ $\boxed{1}$ $\quad$ $b = k_2 B / k_4$ $\quad$ λ $\quad$ $\boxed{1}$ $\quad$ $\tau = 1 / k_4$

$$\dot{u} = \frac{D_u \tau}{\ell^2} \Delta u + \frac{k_1 A \tau}{c} - k_2 B \tau u + k_3 c c' \tau u^2 v - k_4 \tau u$$
$$\dot{v} = \frac{D_v \tau}{\ell^2} \Delta v \qquad + \frac{k_2 B c \tau}{c'} u - k_3 c^2 \tau u^2 v$$

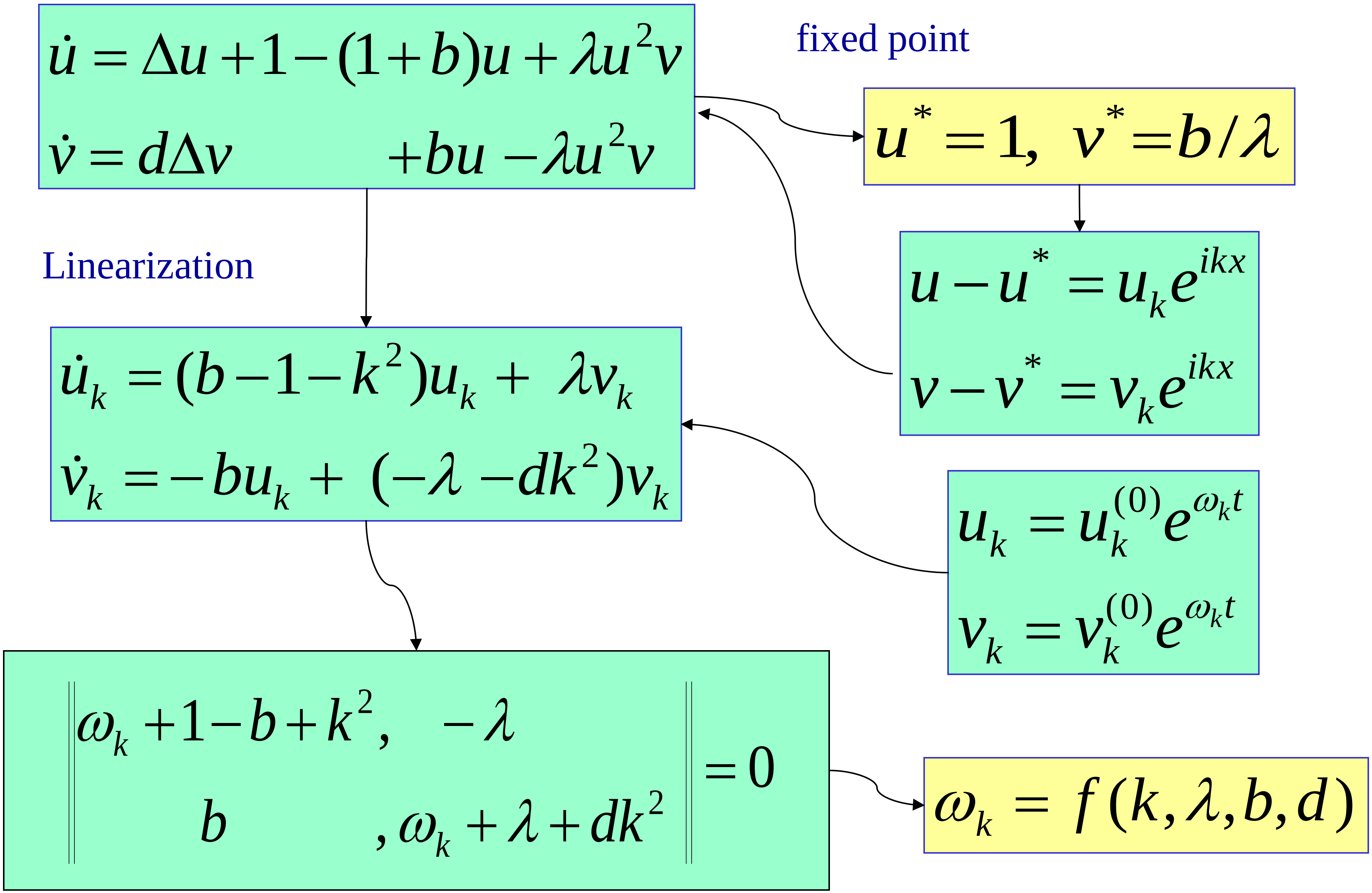
$d = D_v / D_u$

$c = c' = \dfrac{k_1 A}{k_4}$

b

$\lambda = \dfrac{k_1^2 k_3 A^2}{k_4^3}$

Brusselator IV - perturbations at the homogeneous fixed point



$$\dot{u} = \Delta u + 1 - (1+b)u + \lambda u^2 v$$
$$\dot{v} = d\Delta v \qquad\qquad + bu - \lambda u^2 v$$

fixed point

$$u^* = 1, \; v^* = b/\lambda$$

$$u - u^* = u_k e^{ikx}$$
$$v - v^* = v_k e^{ikx}$$

Linearization

$$\dot{u}_k = (b - 1 - k^2)u_k + \lambda v_k$$
$$\dot{v}_k = -bu_k + (-\lambda - dk^2)v_k$$

$$u_k = u_k^{(0)} e^{\omega_k t}$$
$$v_k = v_k^{(0)} e^{\omega_k t}$$

$$\left\| \begin{array}{ll} \omega_k + 1 - b + k^2, & -\lambda \\ b & , \omega_k + \lambda + dk^2 \end{array} \right\| = 0$$

$$\omega_k = f(k, \lambda, b, d)$$

Brusselator V- eigenvalue analysis

$$\omega_k^2 + \underbrace{[1+\lambda-b+(d+1)k^2]}_{-2\alpha_k}\omega_k + \underbrace{b\lambda+(1-b+k^2)(\lambda+dk^2)}_{\beta_k} = 0$$

$$\omega_k = \alpha_k \pm \sqrt{\alpha_k^2-\beta_k}$$

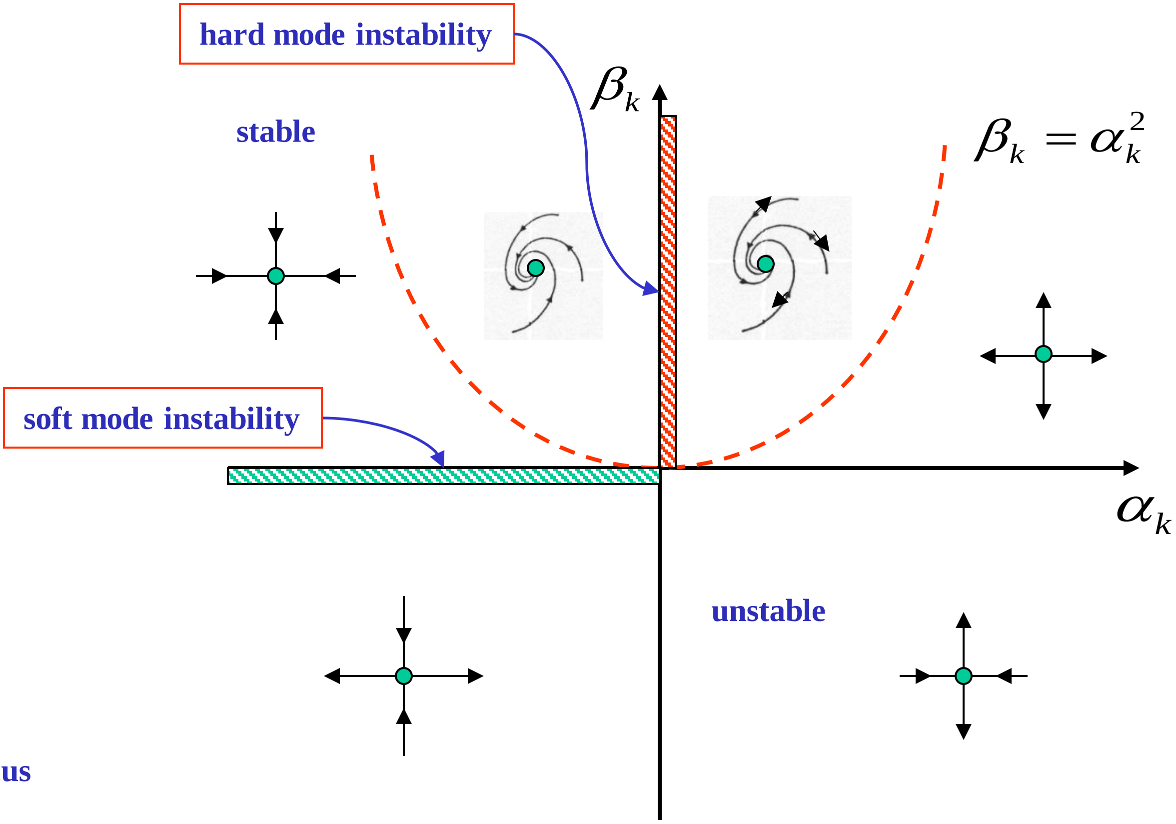
$\beta_k < 0$ ← unstable

$\beta_k > 0,\ \alpha_k > 0$

$\beta_k > 0,\ \alpha_k < 0$ ← stable

$\beta_k > \alpha_k^2$ **stable or unstable focus**

$\alpha_k < 0$ $\alpha_k > 0$



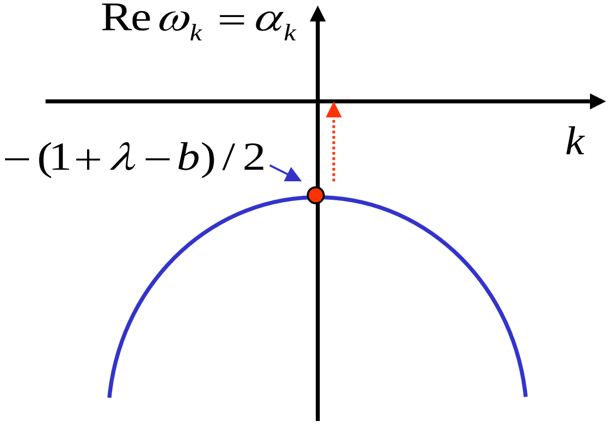
Brusselator VI- hard mode instability

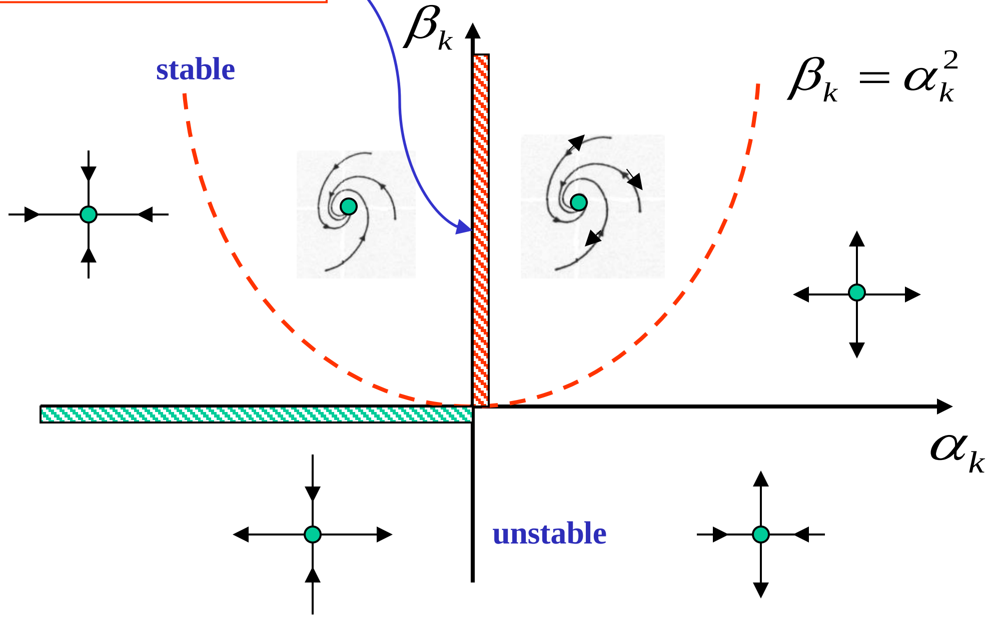
$$\omega_k^2 + \underbrace{[1+\lambda-b+(d+1)k^2]}_{-2\alpha_k}\omega_k + \underbrace{b\lambda+(1-b+k^2)(\lambda+dk^2)}_{\beta_k} = 0$$

$$\omega_k = \alpha_k \pm \sqrt{\alpha_k^2 - \beta_k}$$

$$\beta_k > 0,\ \ \alpha_k \to 0^-$$

hard mode instability





(1) k=0 instability

(2) instability point $\lambda_c = b-1$

(3) exists only for $b > 1$

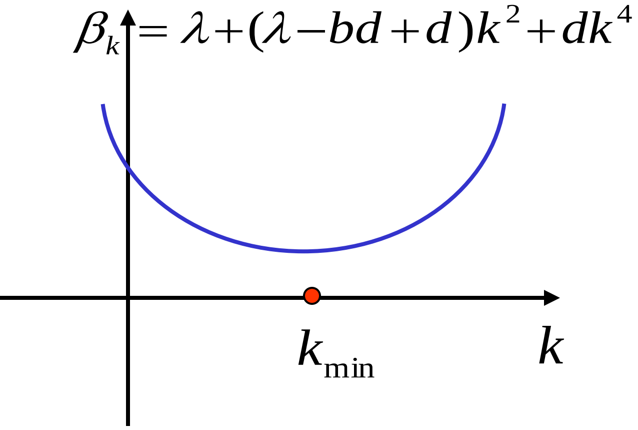
$$\beta_{k=0} = \lambda > 0$$

Brusselator VII- soft mode instability

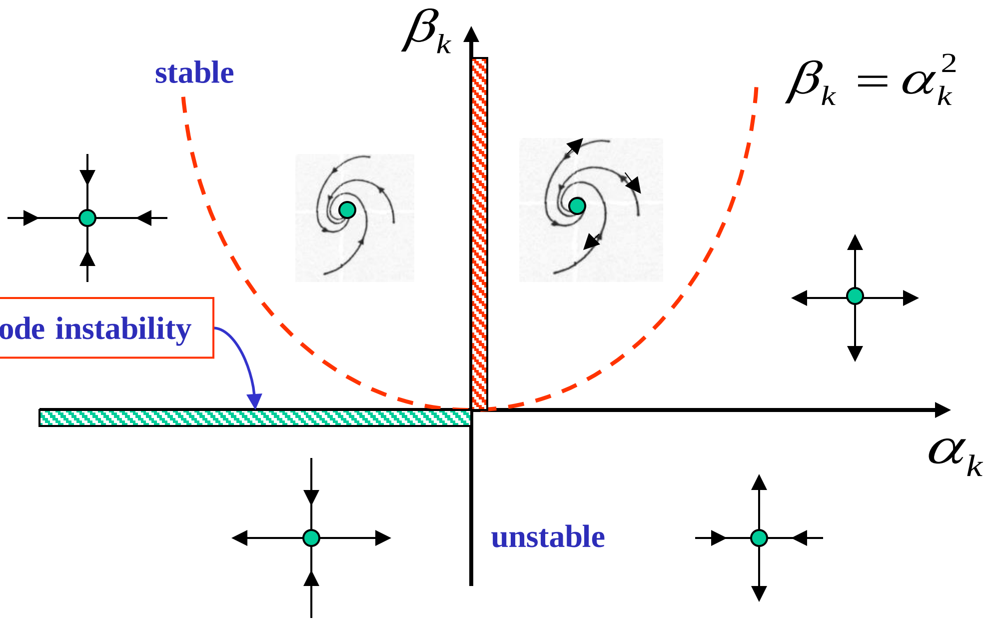
$$\omega_k^2+\underbrace{[1+\lambda-b+(d+1)k^2]}_{-2\alpha_k}\omega_k+\underbrace{b\lambda+(1-b+k^2)(\lambda+dk^2)}_{\beta_k}=0$$

$$\omega_k=\alpha_k\pm\sqrt{\alpha_k^2-\beta_k}$$

$$\alpha_k<0,\ \ \beta_k\to 0^+$$



$$\beta_k=\lambda+(\lambda-bd+d)k^2+dk^4$$



instability point

$$\beta_k=0 \qquad \frac{\partial\beta_k}{\partial k^2}=0$$

$$\lambda_c=d(\sqrt{b}-1)^2$$

Soft mode instabtility first if

$$\lambda_c^{soft}>\lambda_c^{hard} \rightarrow d>\frac{\sqrt{b}+1}{\sqrt{b}-1}>1 \rightarrow D_v>D_u$$

Emergence of spatial structures: Stability analysis

Linearization

$$\dot{n}_1 = D_1 \Delta n_1 + f_1(n_1, n_2, \lambda)$$
$$\dot{n}_2 = D_2 \Delta n_2 + f_2(n_1, n_2, \lambda)$$

$$n_1^*, n_2^*$$

$$n_1 = n_1^* + \delta n_{1k} e^{ikx}$$
$$n_2 = n_2^* + \delta n_{2k} e^{ikx}$$

$$\begin{pmatrix} \delta \dot{n}_{1k} \\ \delta \dot{n}_{2k} \end{pmatrix} = \mathbf{A}_\lambda(k) \begin{pmatrix} \delta n_{1k} \\ \delta n_{2k} \end{pmatrix}$$

Stability matrix

Diagonalization

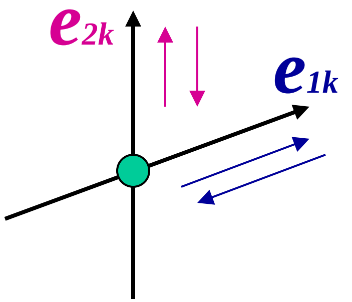
$$\tilde{\mathbf{A}}_\lambda \Rightarrow \begin{pmatrix} \omega_{1\lambda}(k) & 0 \\ 0 & \omega_{2\lambda}(k) \end{pmatrix}$$

Eigenvalues

Solution

$$\begin{pmatrix} \delta n_{1k} \\ \delta n_{2k} \end{pmatrix} = c_{1k} \mathbf{e}_{1k} \, e^{\omega_1(k)t} + c_{2k} \mathbf{e}_{2k} \, e^{\omega_2(k)t}$$

Eigenvectors



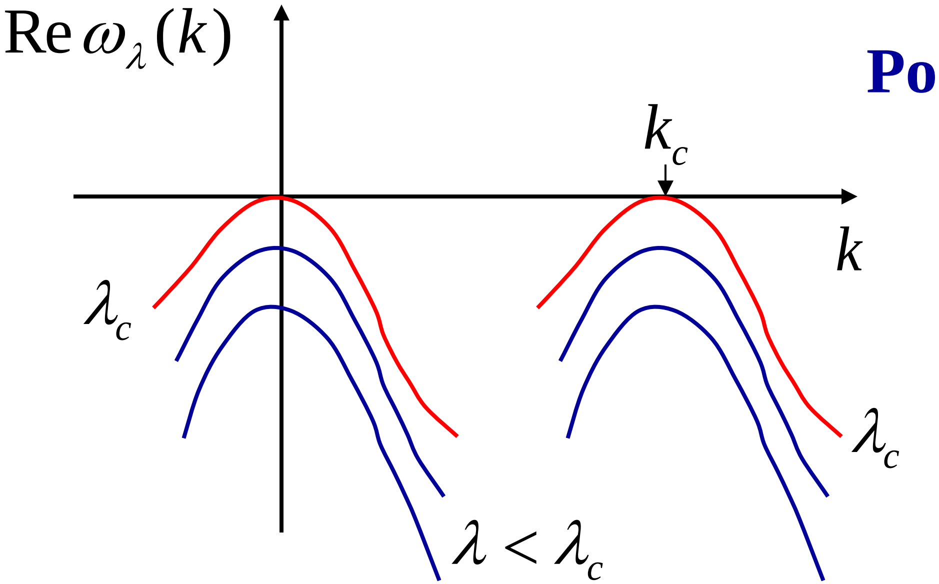
Critical slowing down and classification of instabilities

$$\omega_{\lambda 1,2}(k) \Rightarrow \omega_{\lambda}(k)$$

- with the largest real part

Instability: $$\operatorname{Re}\omega_{\lambda}(k) \rightarrow 0^{-}$$

$$\lambda \rightarrow \lambda_c$$



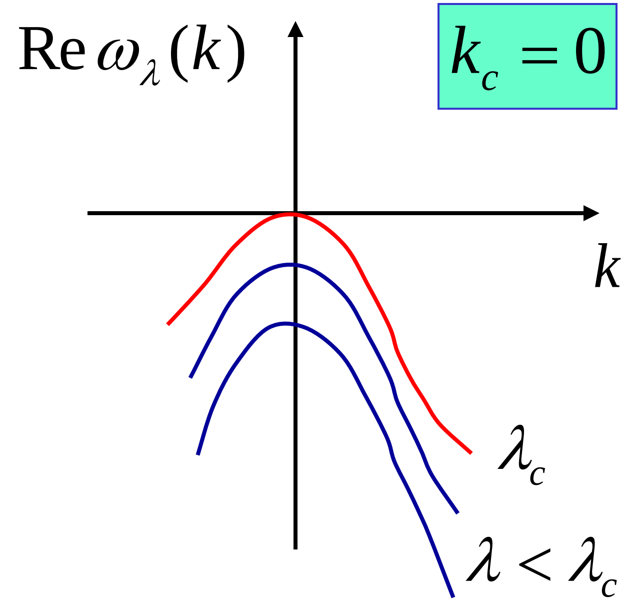
Possibilities:

$$k_c \begin{cases} = 0 \\ \neq 0 \end{cases}$$

$$\operatorname{Im}\omega_{\lambda_c}(k_c) \begin{cases} = 0 & \text{soft} \\ \neq 0 & \text{hard} \end{cases}$$

Classification of instabilities - emerging structures

$k_c = 0$



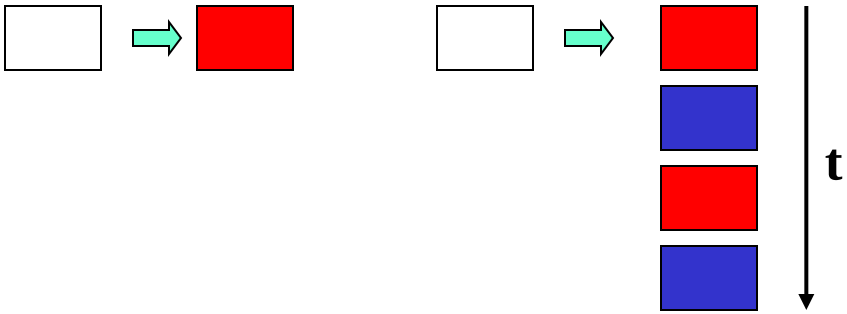
$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$

$\mathrm{Im}\,\omega_{\lambda_c}(k_c) \neq 0$

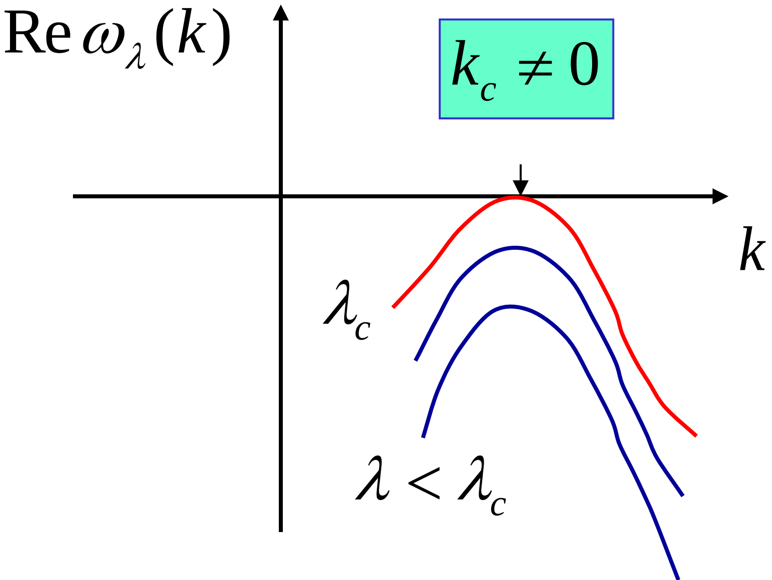
spatially homogeneous

stationary

limit cycle



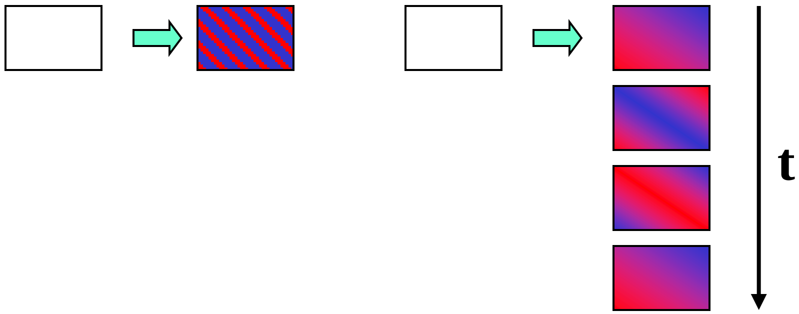
$k_c \neq 0$



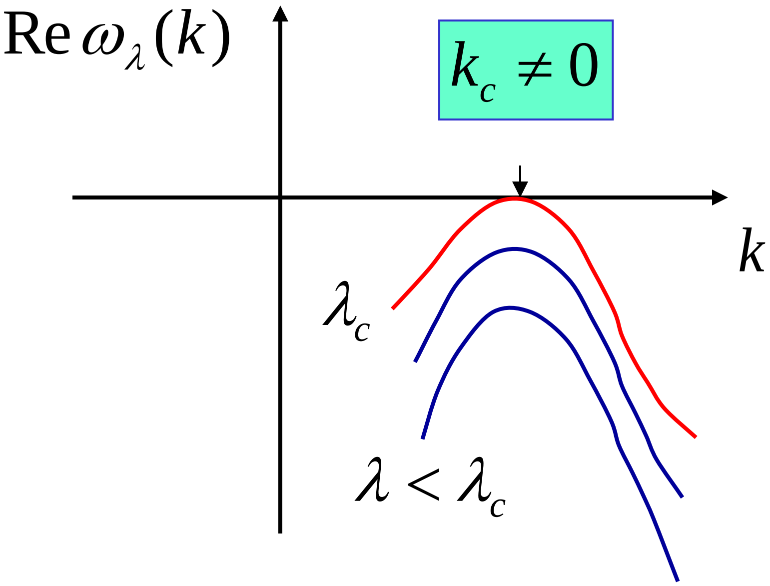
spatially structured

stationary

time- and space-dependent

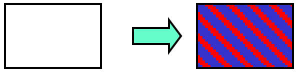


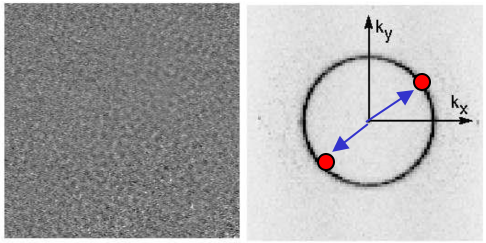
Stationary structures emerging in d=2 homogeneous systems



$$k_c \neq 0$$

$$\mathrm{Im}\,\omega_{\lambda_c}(k_c) = 0$$

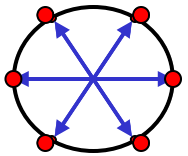


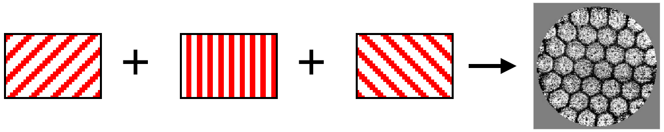


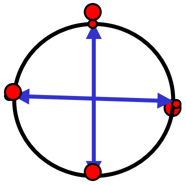
d=2
isotropy

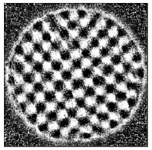
$$\omega_\lambda(\vec{k}) \Rightarrow \omega_\lambda(k)$$

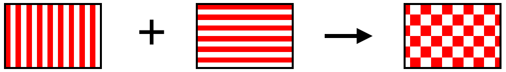
$$\delta n_{\vec{k}} \sim a_{\vec{k}} e^{i\vec{k}\vec{x}+\omega_\lambda(k)t}$$



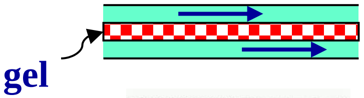


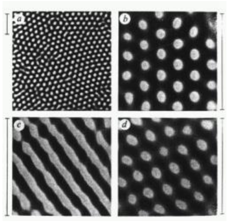




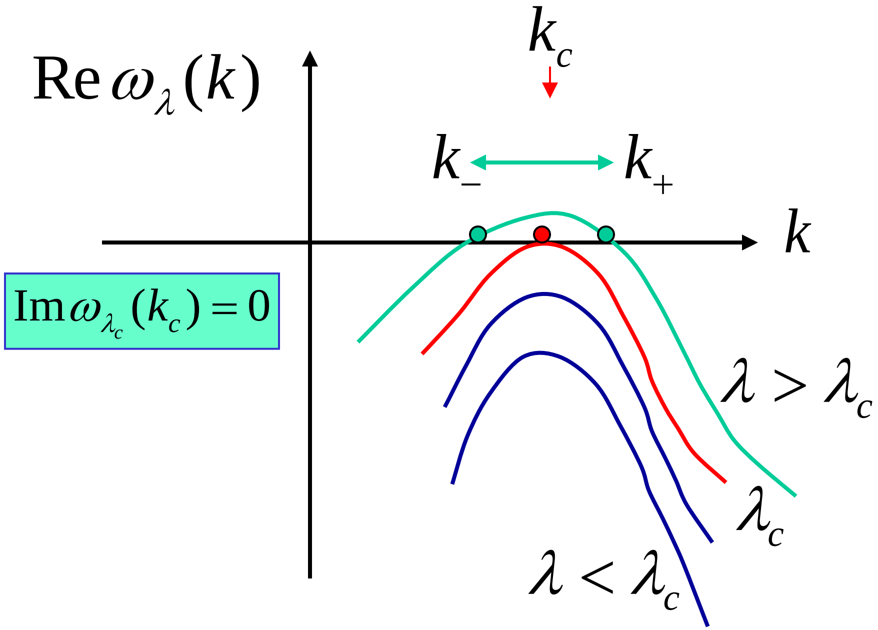


Swinney et al. 1991
Turing patterns



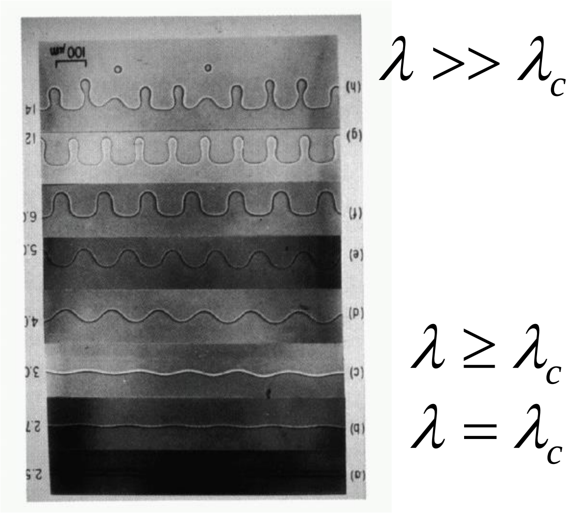


Beyond the instability: Amplitude equation for slow modes



$\lambda > \lambda_c$ Band of unstable modes

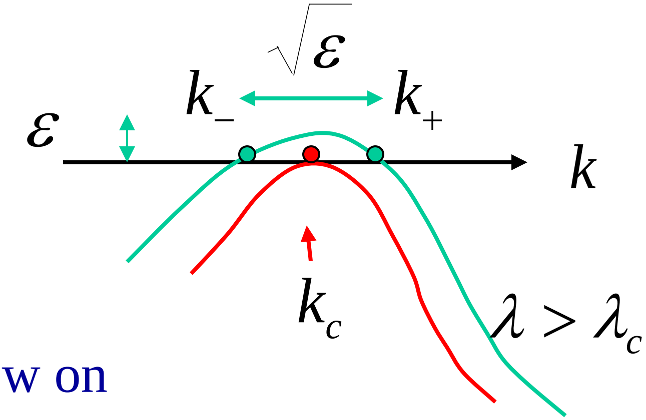
What is the steady state?



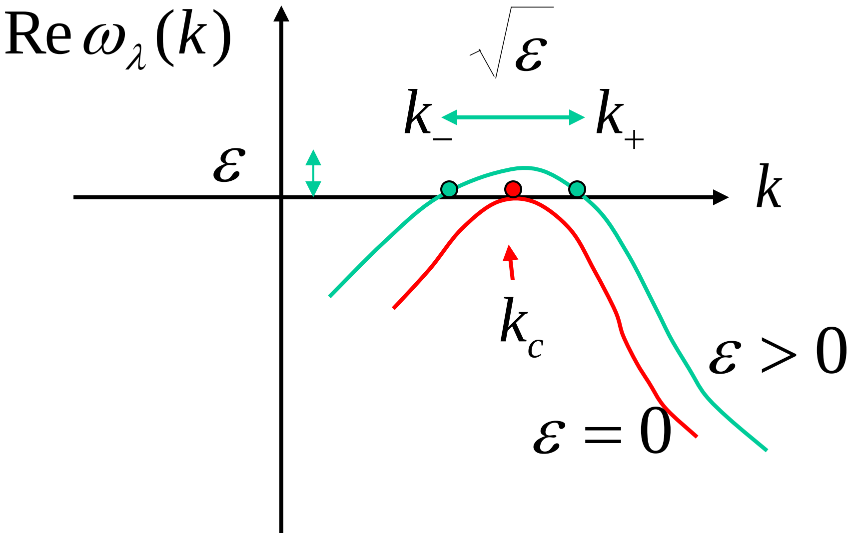
$\mathrm{Re}\,\omega_\lambda(k)$ smooth function of λ and k

$$\boxed{\omega_\lambda(k) \approx \underbrace{\lambda-\lambda_c}_{\boxed{\varepsilon}} - a(k-k_c)^2}$$

control parameter from now on



Amplitude equation: Characteristic lengths and times



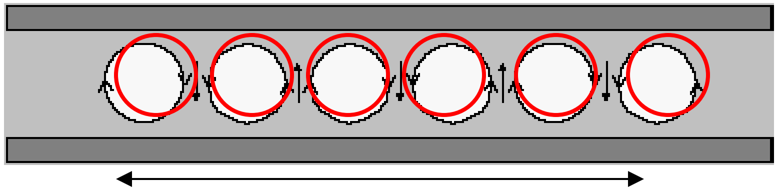
$\varepsilon > 0$ Band of unstable modes

$$\omega_\lambda(k) \approx \varepsilon - a(k-k_c)^2$$

$$n(x,t) - n^* = \int_{-\infty}^{\infty} dk\, \tilde{n}_k\, e^{ikx+\omega_\lambda(k)t}$$

$$\approx e^{ik_c x} \overbrace{\int_{k_-}^{k_+}}^{\sim\sqrt{\varepsilon}} dk\, \tilde{n}_k\, e^{i\overbrace{(k-k_c)}^{\sim\sqrt{\varepsilon}}x + \overbrace{\omega_\lambda(k)t}^{\sim\varepsilon}}$$

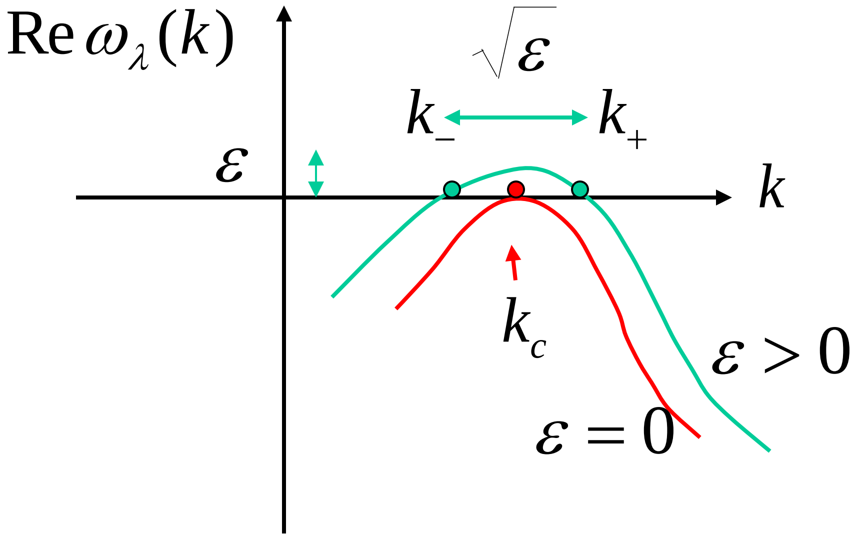
$$\approx e^{ik_c x}\sqrt{\varepsilon}A_0(\sqrt{\varepsilon}x, \varepsilon t)$$



$\xi \sim 1/\sqrt{\varepsilon}$

Variation of the amplitude of the periodic structure on lengthscale $\xi \sim 1/\sqrt{\varepsilon}$ and on timescale $\tau \sim 1/\varepsilon$.

Amplitude equation



$\varepsilon > 0$ Band of unstable modes

$$\boxed{\omega_\lambda(k) \approx \varepsilon - a(k-k_c)^2}$$

$$\boxed{\xi \sim 1/\sqrt{\varepsilon} \qquad \tau \sim 1/\varepsilon}$$

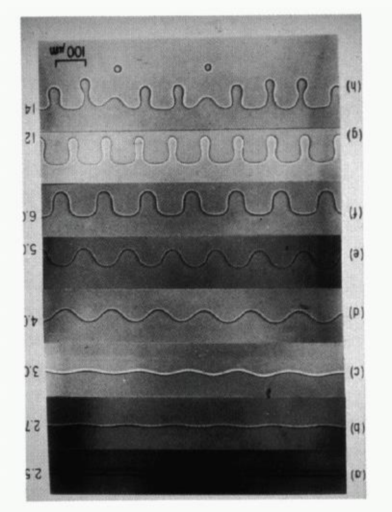
$$\boxed{n(x,t) - n^* \approx e^{ik_c x}\sqrt{\varepsilon}A_0(\sqrt{\varepsilon}x, \varepsilon t) \equiv e^{ik_c x}A(x,t)}$$

Plug it in the original equation and expand.

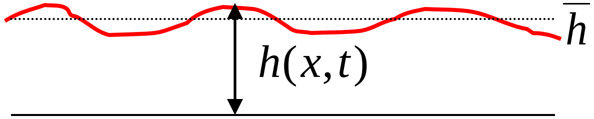
Amplitude equation:

$$\boxed{\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A}$$

Amplitude eq.: Derivation from the Swift-Hohenberg equation



$\varepsilon >> 0$



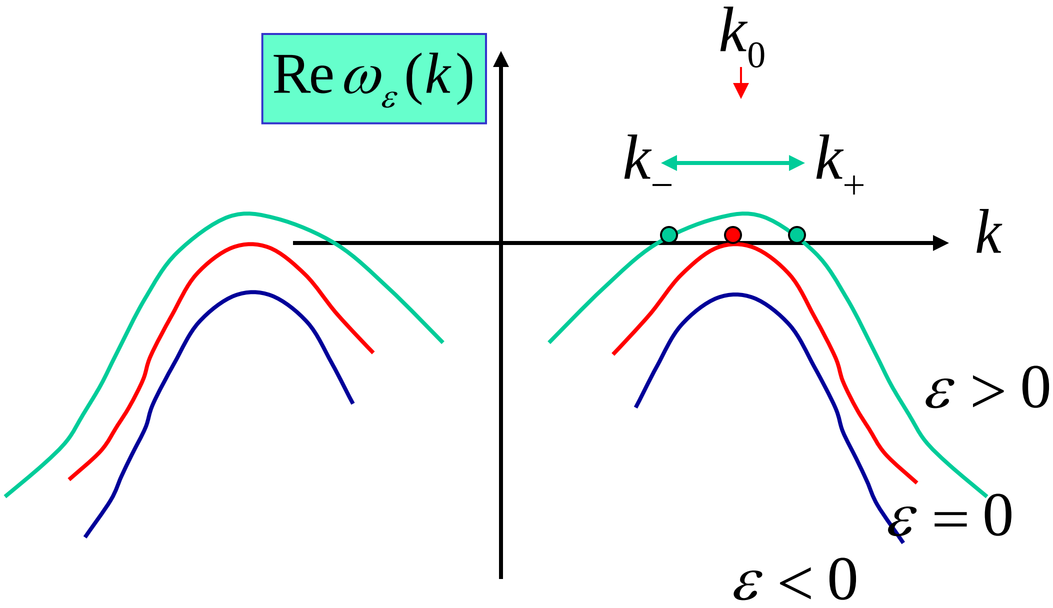
$u(x,t) = h(x,t) - \bar{h}$

$\varepsilon \geq 0$

$\varepsilon = 0$

near instability: $\partial_t u = \varepsilon u - (\Delta + k_0^2)^2 u - u^3$

linearization: $\partial_t u_k = \underbrace{[\varepsilon - (k^2 - k_0^2)^2]}_{\omega_\varepsilon(k)} u_k$



$\mathrm{Im}\,\omega_0(k_0) = 0$

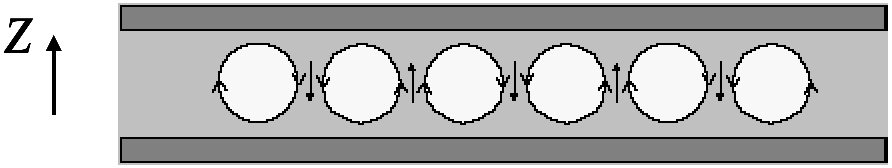
Detailed derivation
in separate pdf file

Amplitude equation: Simple solutions

small $\varepsilon > 0$

$$\boxed{\underbrace{n(x,t)}_{\boxed{v_z}} - n^* \approx e^{ik_c x}\sqrt{\varepsilon}\,A_0(\sqrt{\varepsilon}\,x, \varepsilon t) \equiv e^{ik_c x} A(x,t)}$$

z-component of the velocity



$$\boxed{v_z = A\cos(k_c x)}$$

$A =$ const.

$$\cancel{\frac{\partial A}{\partial t}} = \varepsilon A + \cancel{\frac{\partial^2 A}{\partial x^2}} - |A|^2 A \longrightarrow A = \pm\sqrt{\varepsilon}$$

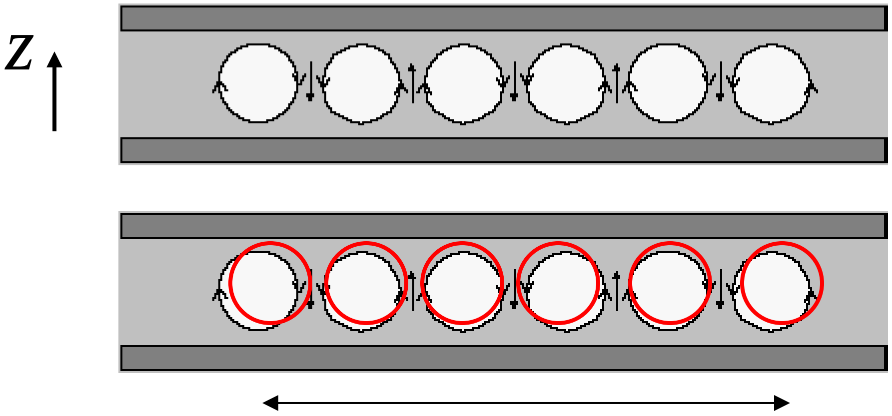
$$\boxed{\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A} \xrightarrow[\text{general solution}]{} \boxed{A = \sqrt{\varepsilon}\,A_0(\sqrt{\varepsilon}\,x, \varepsilon t)}$$

Amplitude equation: Why is it so general?

small $\varepsilon > 0$

$$\underbrace{n(x,t)-n^*}_{\mathrm{v}_z} \approx e^{ik_c x}\sqrt{\varepsilon}A_0(\sqrt{\varepsilon}x,\varepsilon t) \equiv e^{ik_c x}A(x,t)$$

z-component of the velocity



$$\xi \sim 1/\sqrt{\varepsilon}$$

Linear stability changes with ε changing sign

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

lowest order in spatial derivatives preserving ↔ symmetry

Slow, large-scale motion around the stationary structure should not depend on the position of the underlying structure

$$e^{ik_c x}A(x,t) \rightarrow e^{ik_c(x+l)}A(x,t)$$

$$= e^{ik_c x}e^{ik_c l}A(x,t) \rightarrow e^{ik_c x}B(x,t)$$

Amplitude equation: What can we get out of it?

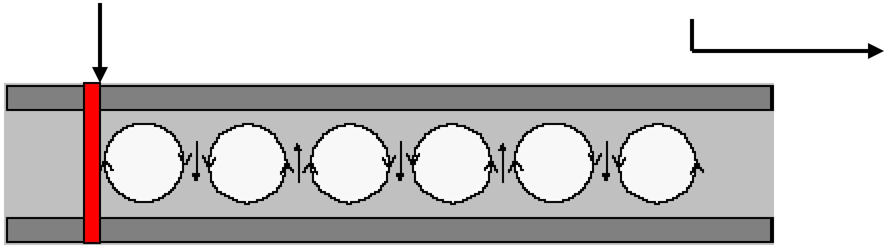
$$\mathrm{v}_z = e^{ik_c x} A(x,t)$$

$$\varepsilon > 0$$

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

$$\mathrm{v}_z = 0$$

$$\mathrm{v}_z = \sqrt{\varepsilon} e^{ik_c x}$$



stationary state

$$A = 0 \longleftrightarrow A(x \to \infty) = \sqrt{\varepsilon}$$

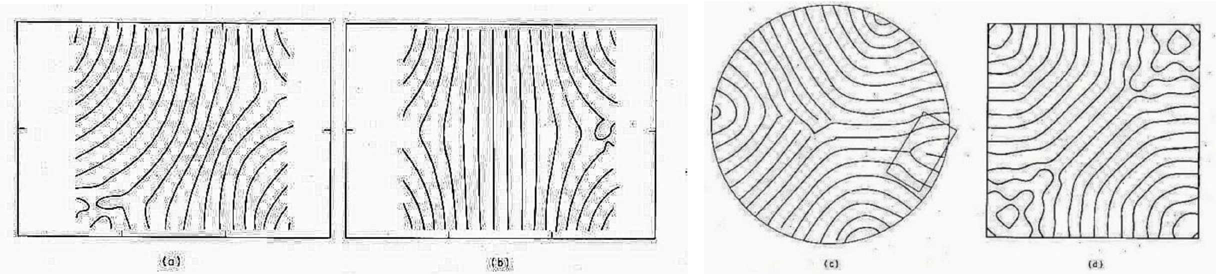
$$\xi \sim 1/\sqrt{\varepsilon}$$

boundary conditions

$$0 = \varepsilon A + \frac{d^2 A}{d x^2} - |A|^2 A$$

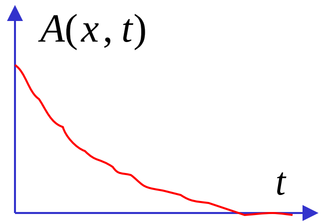
$$A = \sqrt{\varepsilon}\,\mathrm{th}\left(\sqrt{\frac{\varepsilon}{2}}\, x\right)$$

Scale of A and x are determined

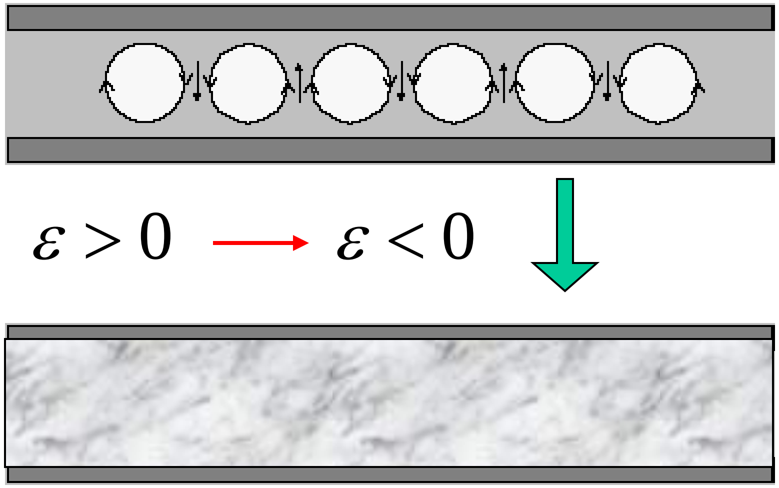


Amplitude equation: Fixing the time-scale

Quenching from ordered into disordered state



$$v_z = e^{ik_c x} A(x,t) \rightarrow 0$$



$\varepsilon > 0 \longrightarrow \varepsilon < 0$

Amplitude equation should be still good

$$\frac{\partial A}{\partial t} = -|\varepsilon| A + \frac{\partial^2 A}{\partial x^2} - \underset{\uparrow}{|A|^2} A$$

$$(A \rightarrow 0) \qquad \approx 0$$

$$A(x,t) = a_k(t) e^{ikx}$$

$$\dot{a}_k = -(\varepsilon + k^2) a_k$$

Relaxation time

$$\tau_k = \frac{1}{\varepsilon + k^2}$$

Amplitude equation: Secondary instabilities I

$\varepsilon > 0$

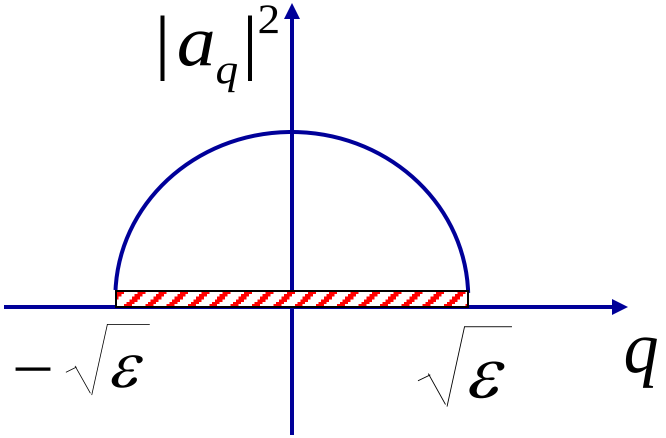
Large number of possible stationary states

$$A = a_q e^{iqx}$$

$$0 = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

$$\left(\varepsilon - q^2 - |a_q|^2\right) a_q = 0$$

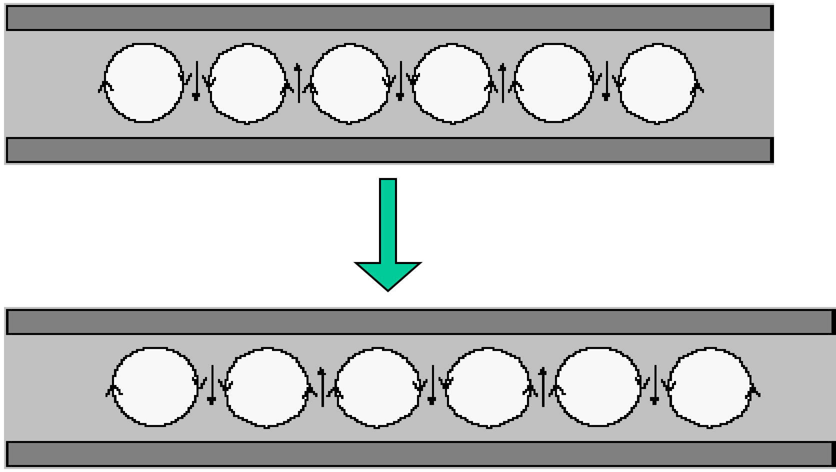
$$|a_q|^2 = \varepsilon - q^2$$



Meaning: Shift in the wavelength of the pattern

$$\mathrm{v}_z = e^{ik_c x} a_0$$

$$\mathrm{v}_z = e^{ik_c x} A = e^{i(k_c+q)x} a_q$$



Amplitude equation: Secondary instabilities II

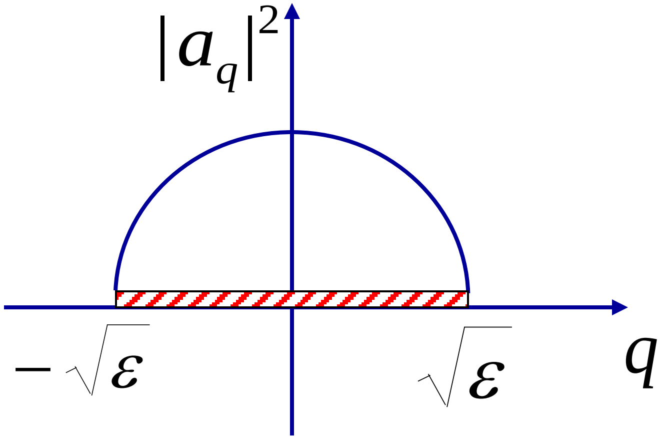
$$\varepsilon > 0$$

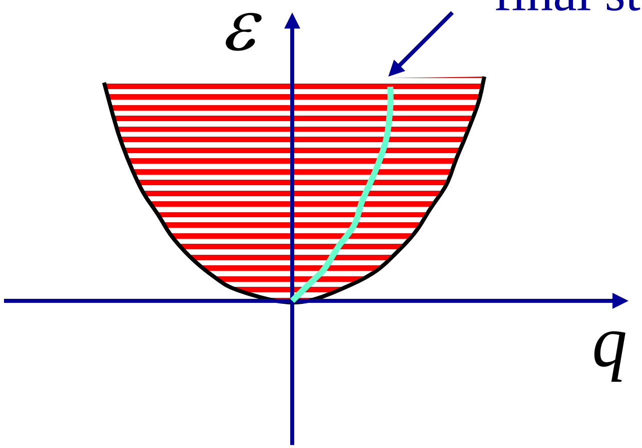
Large number of possible stationary states

$$0 = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$

$$A = a_q e^{iqx}$$

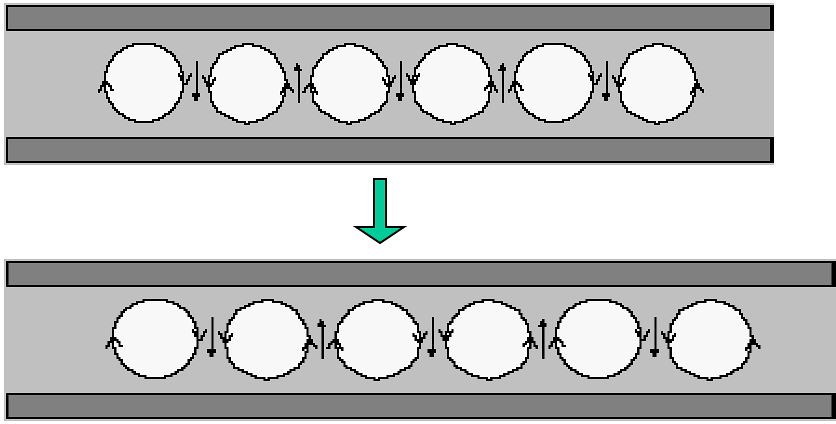
$$v_z = e^{ik_c x} A = e^{i(k_c+q)x} a_q$$



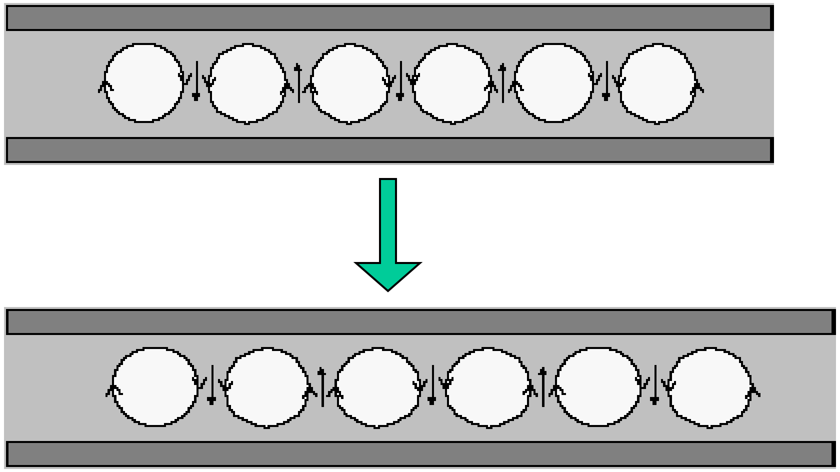


final state

Phase winding solutions



Amplitude equation: Secondary instabilities III



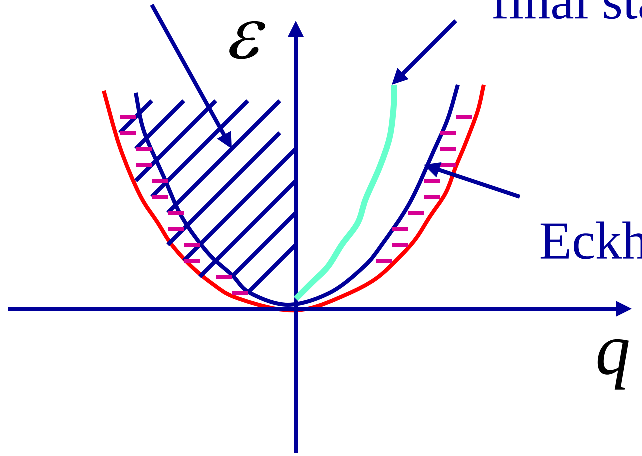
$$v_z = e^{ik_c x} A = e^{i(k_c+q)x} a_q$$

Stability analysis:

$$A = (a_q + \delta a) e^{iqx}$$

$$\delta a = \delta a(x, y, t)$$

$$\frac{\partial A}{\partial t} = \varepsilon A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A$$



Zig-zag instability

final state (?)

Eckhaus instability line

Amplitude eq.: Secondary instabilities: Phase diffusion

Y. Pomeau, P. Manneville

$$\mathrm{v}_z = e^{ik_c x} A$$

$$A = (a_q + \delta a) e^{i(qx+\varphi)}$$

$\varphi = const$

does not decay

decays on timescale

$$\tau_a \sim 1/\varepsilon$$

$\varphi \sim b \sin Qx$

decays as

$$\tau_Q \sim 1/Q^2$$

$$\tau_a << \tau_Q$$

$$\delta a = f(\varphi, \partial_x \varphi)$$

Stability analysis:

$$\partial_t A = \varepsilon A + \partial_x^2 A - \left|A\right|^2 A$$

$$\partial_t \varphi = \frac{\varepsilon - 3q^2}{\varepsilon - q^2} \partial_x^2 \varphi$$

Eckhaus instability line: phase diffusion becomes unstable:

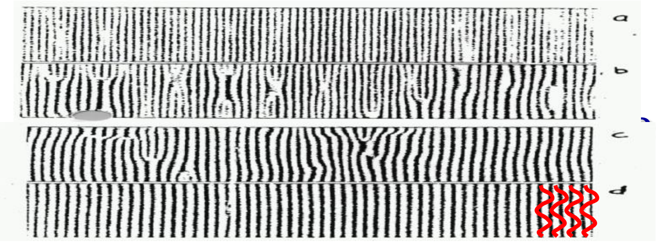
$$q_\pm = \sqrt{\varepsilon / 3}$$

Amplitude equation for A(x,y,t): Secondary instabilities

$$\mathrm{v}_z = e^{ik_c x} A(x,y,t)$$

$$u = \varepsilon^{1/2} A_0(\varepsilon^{1/2}x, \varepsilon^{1/4}y, \varepsilon t)\Phi(x) + \ldots$$

$$\partial_t A = \varepsilon A + \left(\partial_x + \frac{i}{2k_c}\partial_y\right)^2 A - |A|^2 A$$



Stability analysis:

$$A = (a_q + \delta a)e^{i(qx+\varphi)}$$

$$\varphi = \varphi(x,y,t)$$

$$\delta a = f(\varphi, \partial_x \varphi)$$

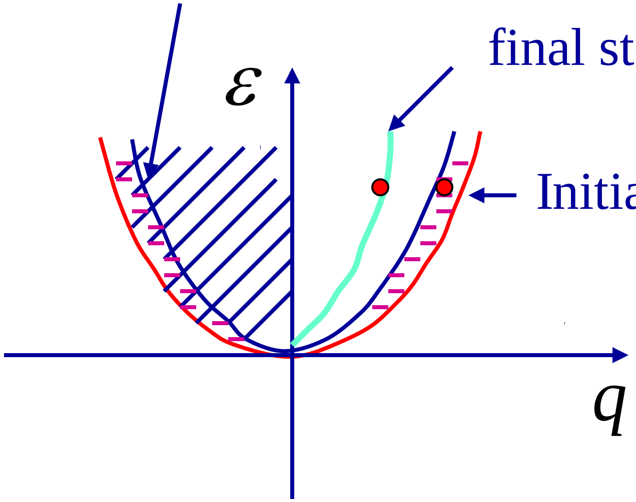
$$\partial_t \varphi = \frac{\varepsilon - 3q^2}{\varepsilon - q^2}\partial_x^2 \varphi + \frac{q}{2k_c}\partial_y^2 \varphi$$

Zigzag instability: $q < 0$

Dynamics of secondary instabilities: Topological defects

$$\mathrm{v}_z = e^{ik_c x} A$$

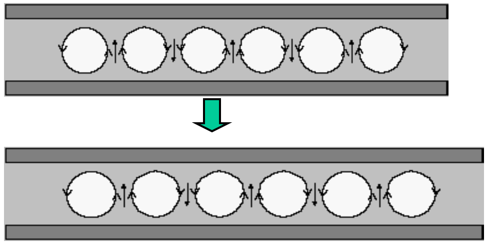
$$A = (a_q + \delta a)e^{i(qx+\varphi)}$$

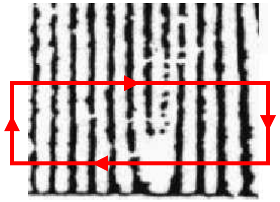


Eckhaus instability line

final state (?)

Initial state

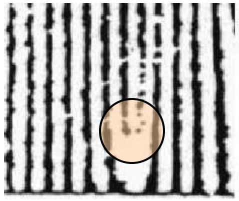


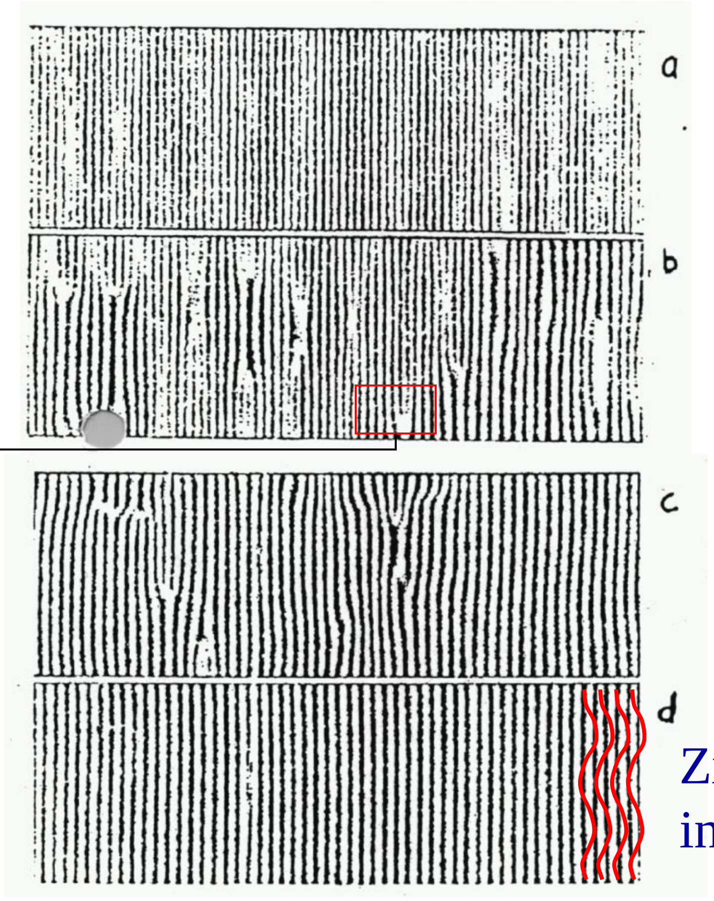


$\delta\varphi = 2\pi$

Not consistent with smooth change

$$\boxed{A \rightarrow 0}$$





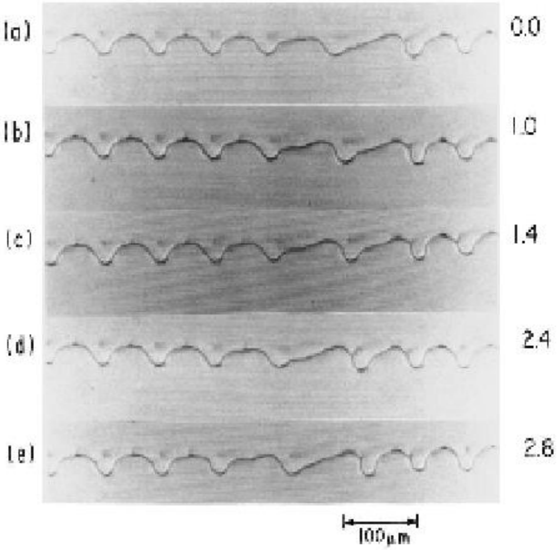
time

Zig-zag instab.

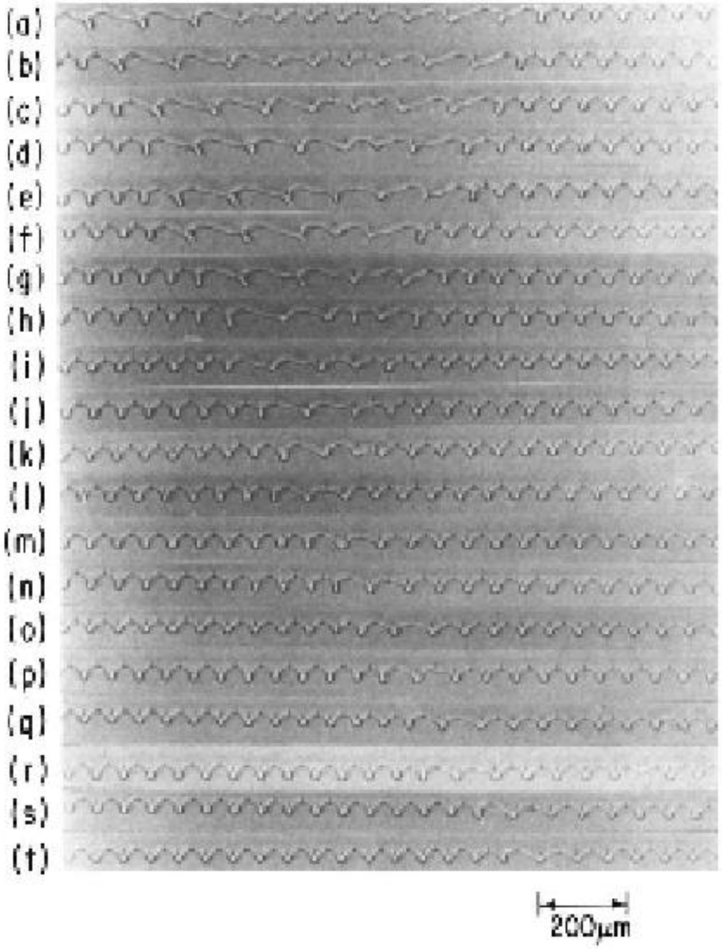
Translating structures

A.J. Simon, J. Bechofer, A. Libchaber

Isotropic-nematic transition



Solitary wave to
moving to the left



Collision of two solitary waves

$$n(x,t)-n^* \approx e^{i(k_c x-\omega_c t)}\sqrt{\varepsilon}A_0(\sqrt{\varepsilon}x,\varepsilon t) \equiv A(x,t)\Phi(k_c x-\omega_c t)$$

Complex Landau-Ginzburg equation

$$k_c \neq 0, \quad \mathrm{Im}\,\omega_{\lambda_c}(k_c) \neq 0$$

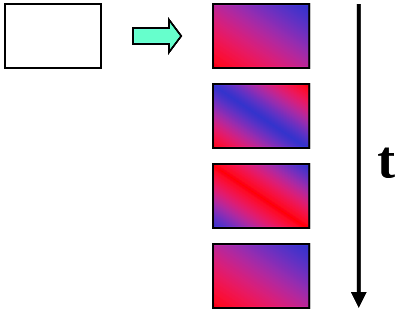
$$n(x,t) - n^* \approx e^{i[k_c x + \mathrm{Im}\,\omega(k_c)t]} A(x,t)$$

$$\frac{\partial A}{\partial t} + v\frac{\partial A}{\partial x} = \varepsilon A + (1 + ic_1)\frac{\partial^2 A}{\partial x^2} - (1 - ic_3)|A|^2 A$$

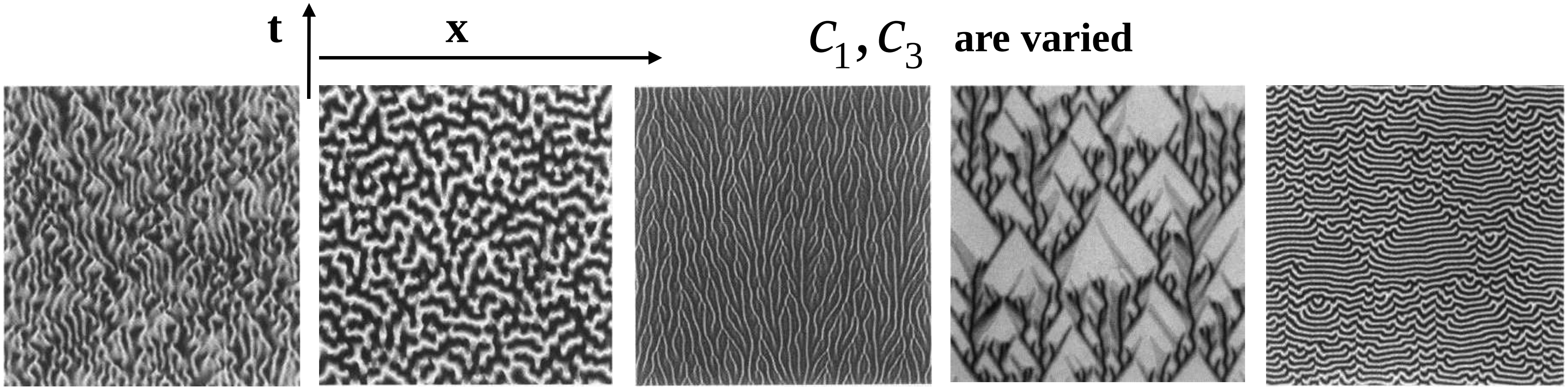
$$v = \mathrm{Im}\,\omega(k_c)/k_c$$

Velocity of the wave

time- and space-dependent



C_1, C_3 **are varied**



H. Chate'

CLG equation: Secondary Instabilities

$$\partial_t A = \varepsilon A + (1 + ic_1)\partial_x^2 A - (1 - ic_3)|A|^2 A$$

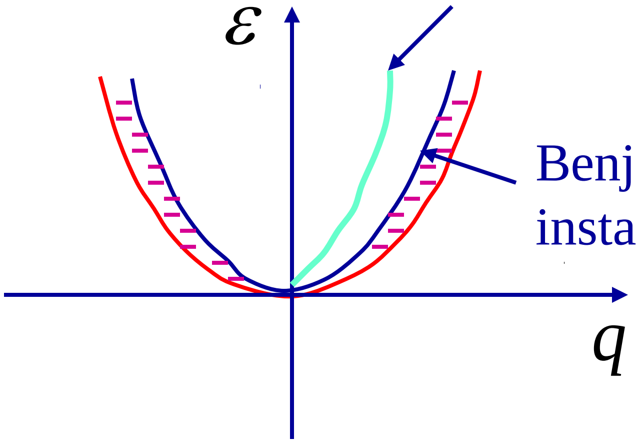
Phase winding solutions

$$A = a_{\omega,q} e^{i(qx-\omega t)}$$

$$\omega = c_1 q^2 - c_3 |a|^2$$

$$q^2 = \varepsilon - |a|^2$$

Linear stability



Benjamin-Feir instability

c_1, c_3 increased ⇨

linearly stable region decreases

$$\boxed{c_1 c_3 > 1}$$

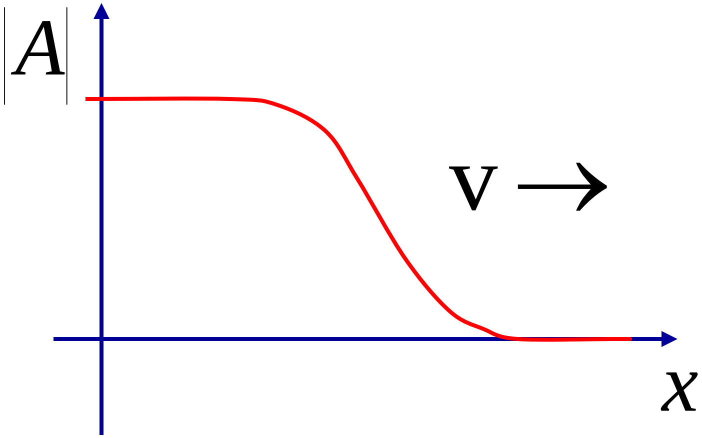
Newell criterion

No linerly stable region exists.

CLG equation: Coherent structures

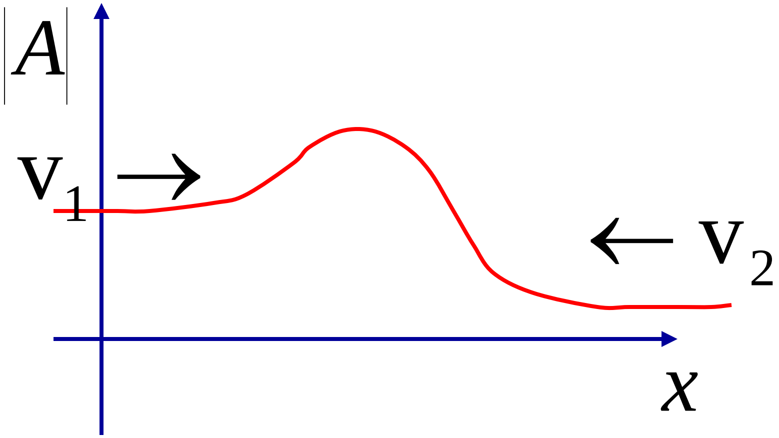
$$\partial_t A = \varepsilon A + (1+ic_1)\partial_x^2 A - (1-ic_3)\left|A\right|^2 A$$

Front



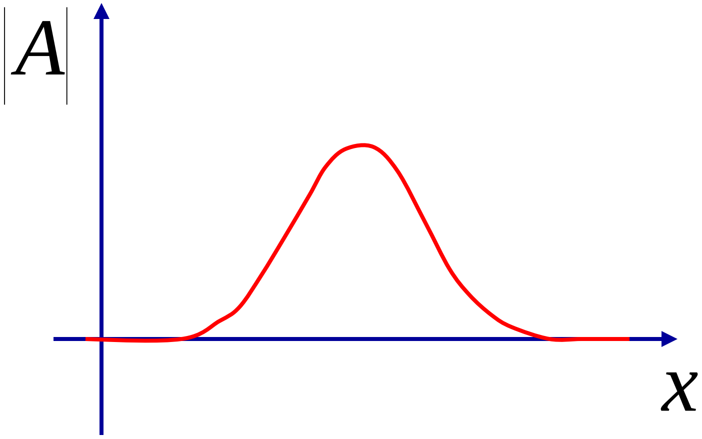
V = ? Pattern left behind?

Sink

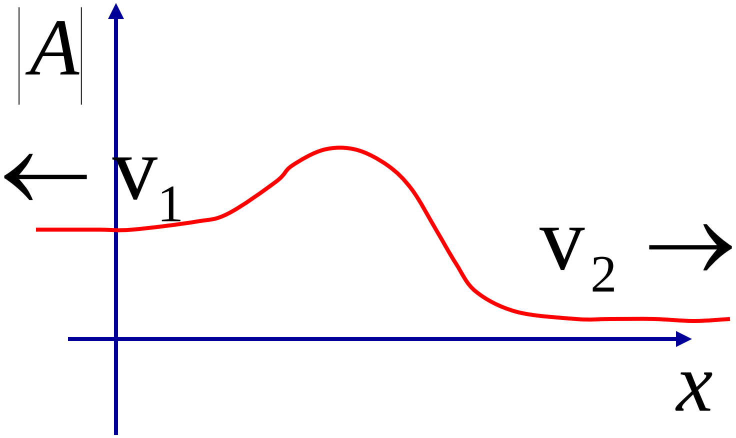


Two different phase-winding solutions

Pulse



Source



CLGE - Phase diagram

$$\frac{\partial A}{\partial t} = \varepsilon A + (1 + ic_1)\frac{\partial^2 A}{\partial x^2} - (1 - ic_3)|A|^2 A$$