Dynamical Approach for Solving Complex Networks

Sidney Redner, Boston University, physics.bu.edu/~redner
collaborators: P. Krapivsky, F. Leyvraz

MECO34: 34th Conference of the Middle European Cooperation in Statistical Physics
30 March - 01 April 2009 Universität Leipzig, Germany

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Outline:

preliminaries: 1. Erdős-Rényi Random Graph
2. Random Recursive Tree (RRT)

linear preferential attachment by *redirection*

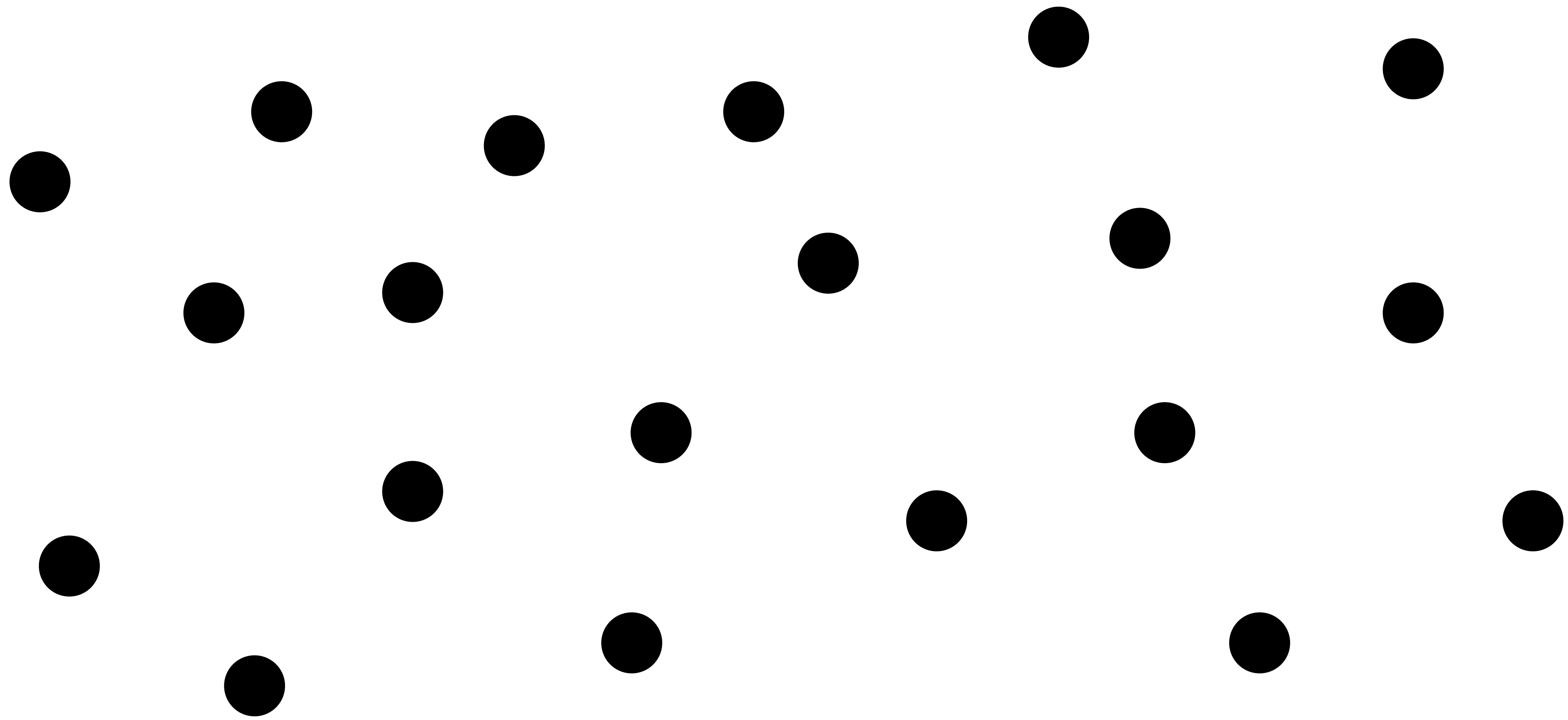
universality classes of preferential attachment

genealogy

outlook

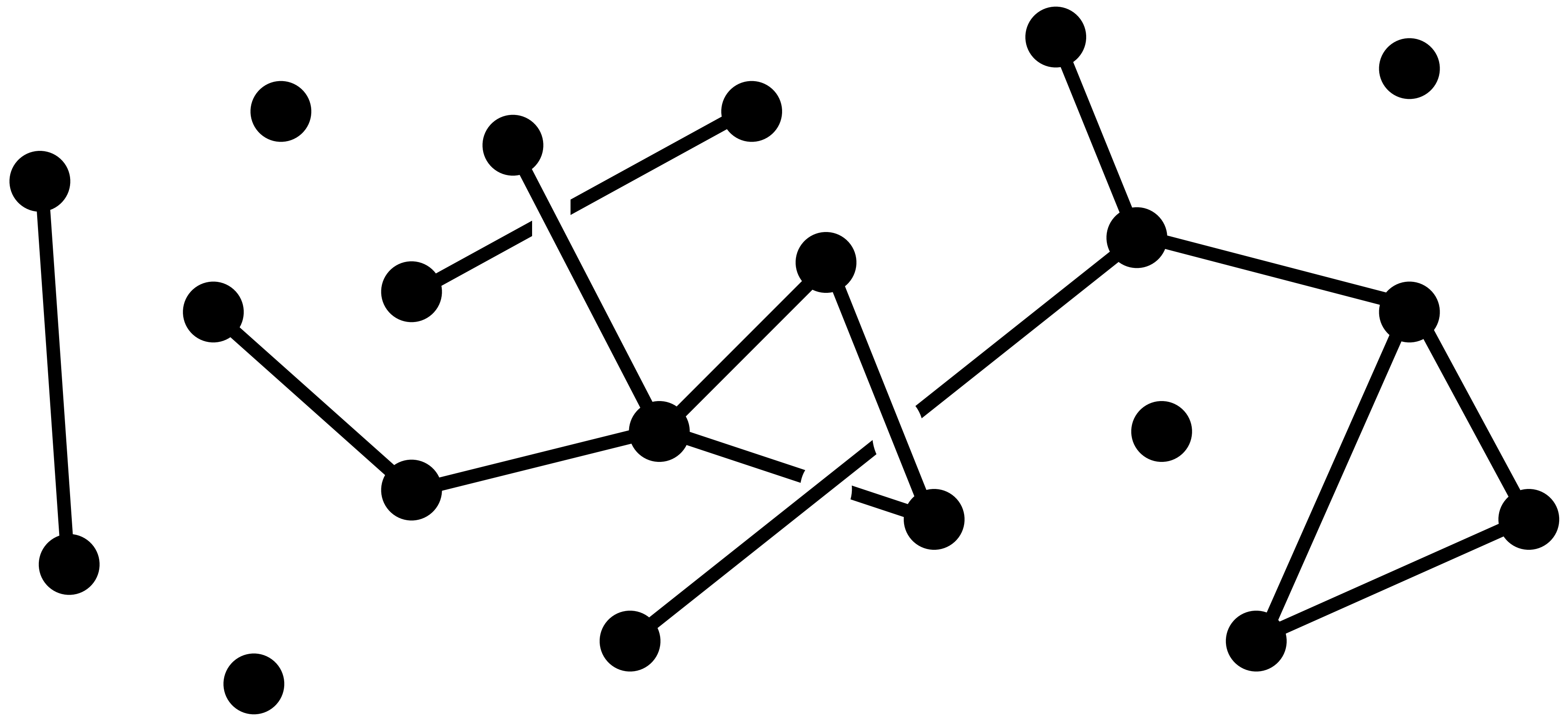
Erdős-Rényi Random Graph

Erdős-Rényi Random Graph



1. Start with N isolated nodes (and $N^2/2$ pairs)

Erdős-Rényi Random Graph



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2. Introduce links at constant rate

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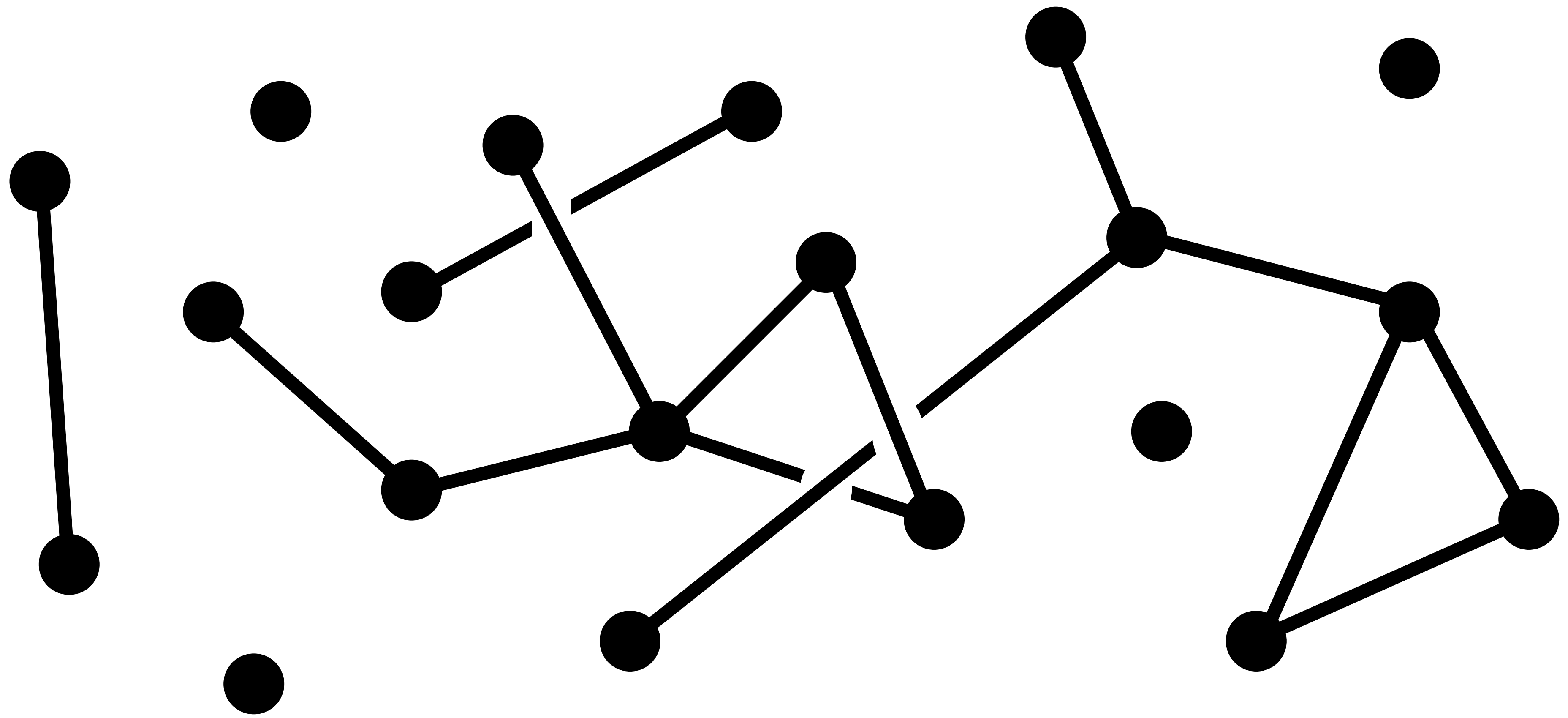

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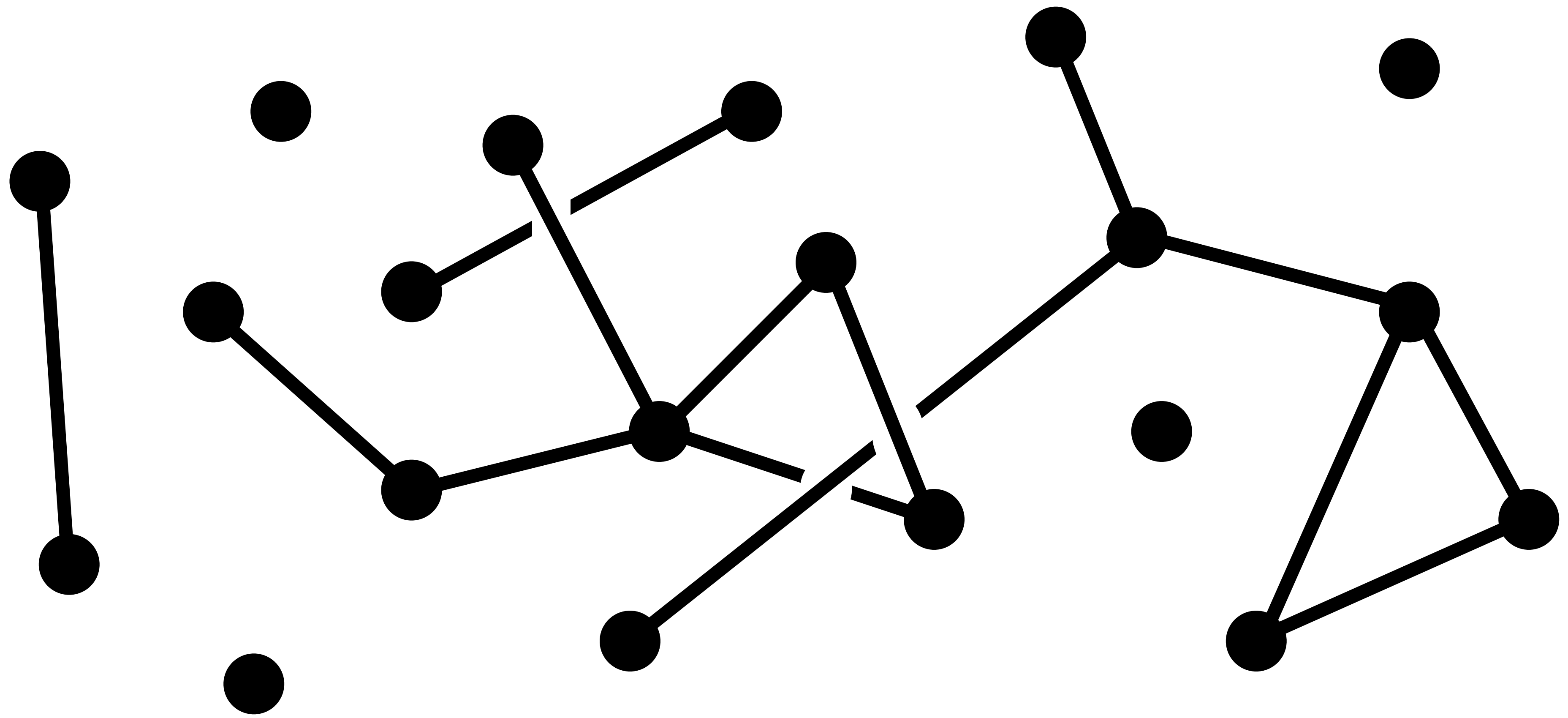
2. Introduce links at constant rate N/2

Erdős-Rényi Random Graph



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Number of links L at time t: $L = N/2 \times t = Nt/2$

Average degree $D = 2L/N = t$

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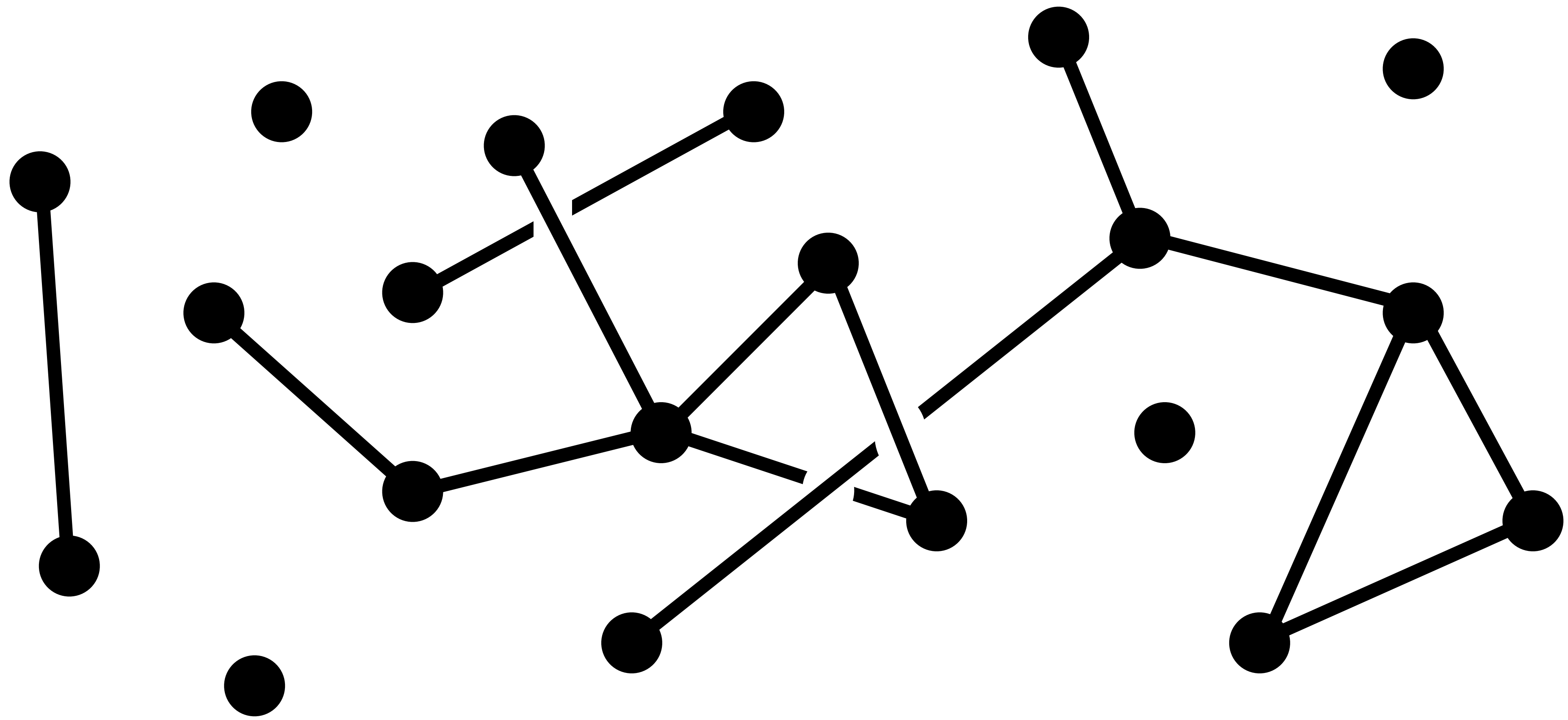

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Number of links L at time t: $L = N/2 \times t = Nt/2$

Average degree $D = 2L/N = t$

Average degree $k \rightarrow k+1$ at rate 1

The Degree Distribution

$$N_k = \text{number of nodes of degree } k$$
$$n_k = \text{fraction of nodes of degree } k$$

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Master equation for n_k

$$\frac{dn_k}{dt} = n_{k-1} - n_k$$

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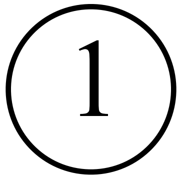
Solution

$$n_k = \frac{t^k}{k!} e^{-kt}$$

Poisson degree distribution

Random Recursive Tree

Random Recursive Tree



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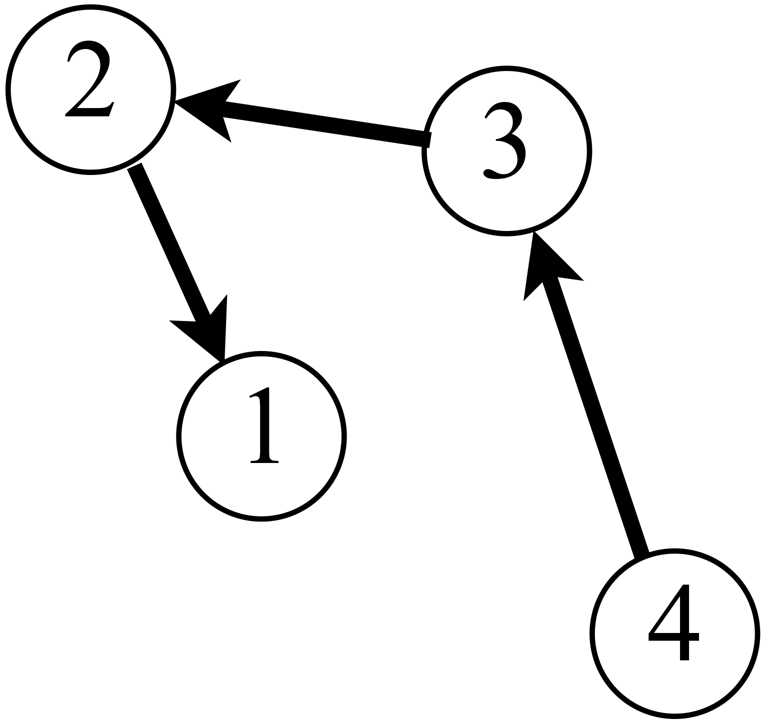
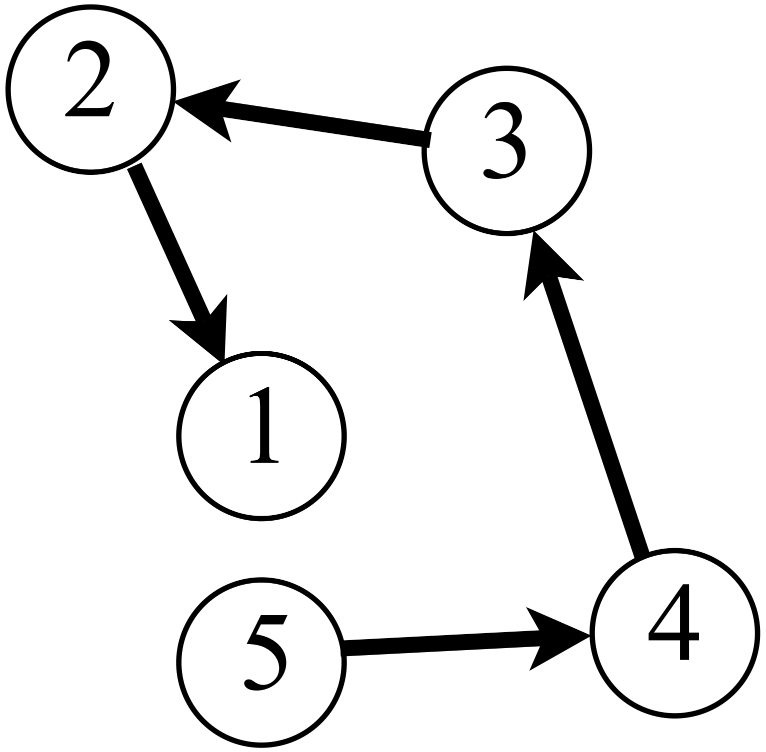


1. Introduce nodes one at a time

Random Recursive Tree



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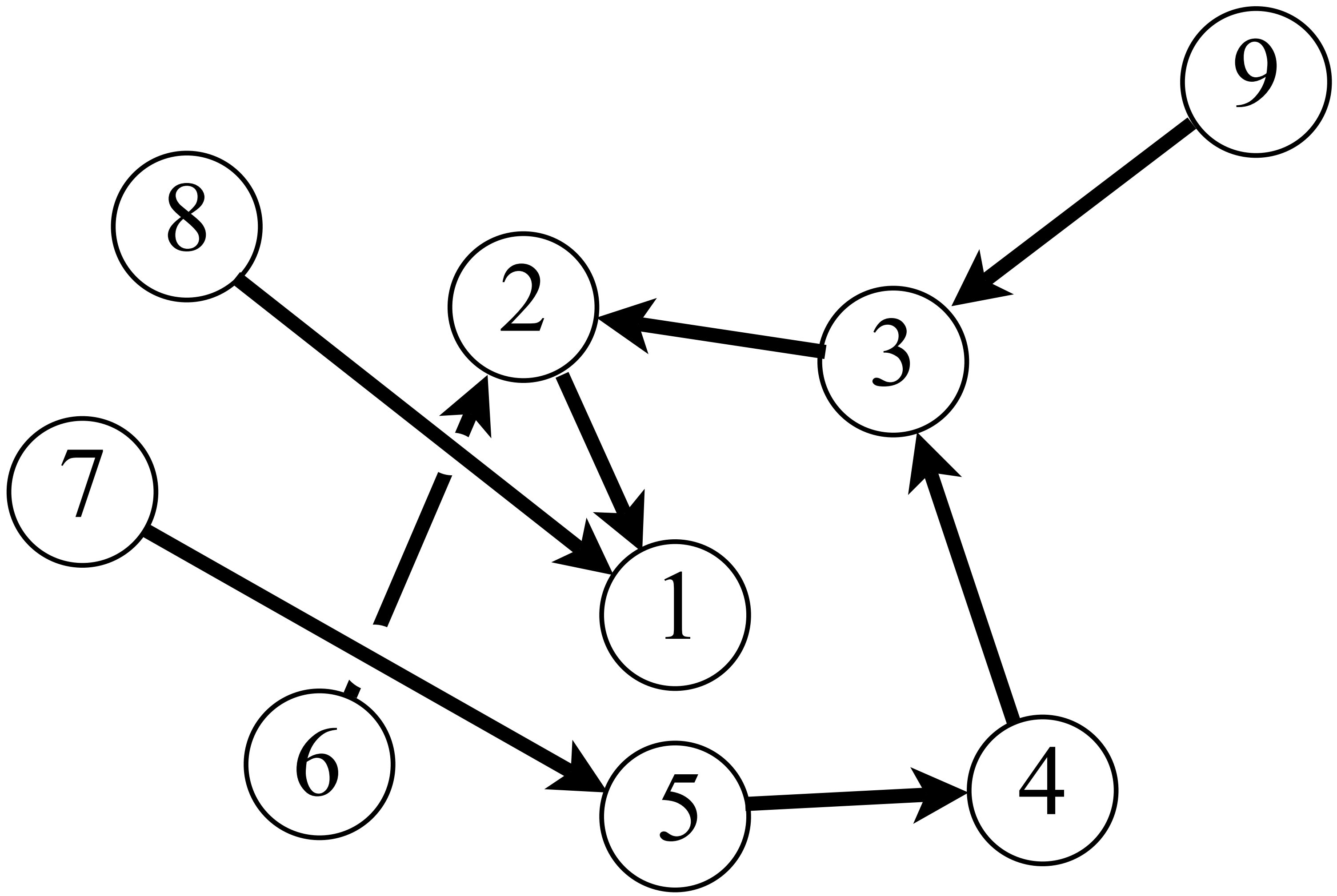

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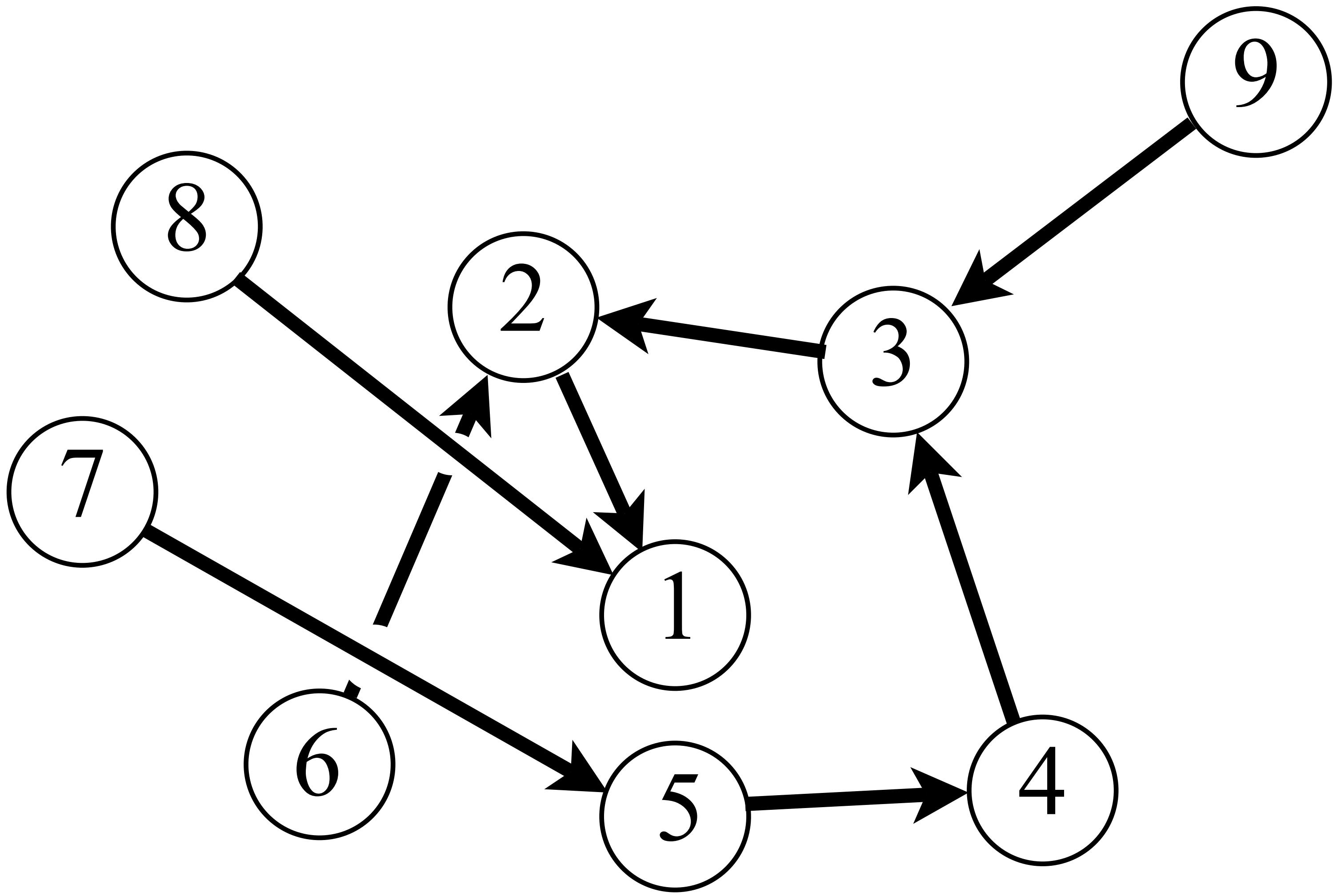

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Basic observable: N_k, the degree distribution

The Degree Distribution

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Master Equation

$$\frac{dN_k}{dN} = \frac{N_{k-1} - N_k}{N} + \delta_{k,1}$$

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"time" variable N

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solving one by one:

$$N_0 = \frac{1}{N},\ N_1 = \frac{N}{2},\ N_2 = \frac{N}{4},\ldots$$

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solution: $n_k = 2^{-k}$

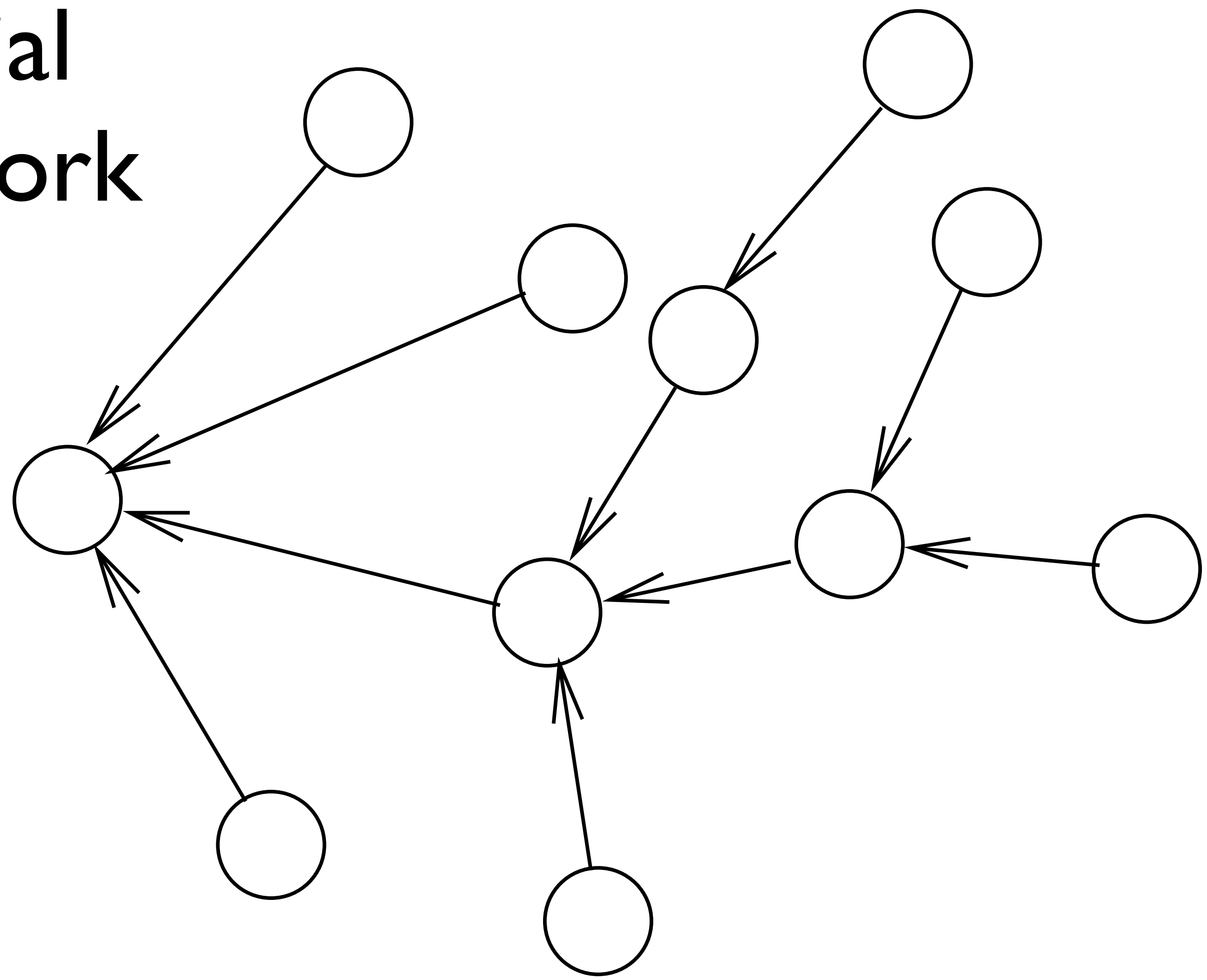
Uniform Attachment + *Redirection*

Krapivsky &
SR (2001)

Uniform Attachment + *Redirection*

Krapivsky & SR (2001)

initial network



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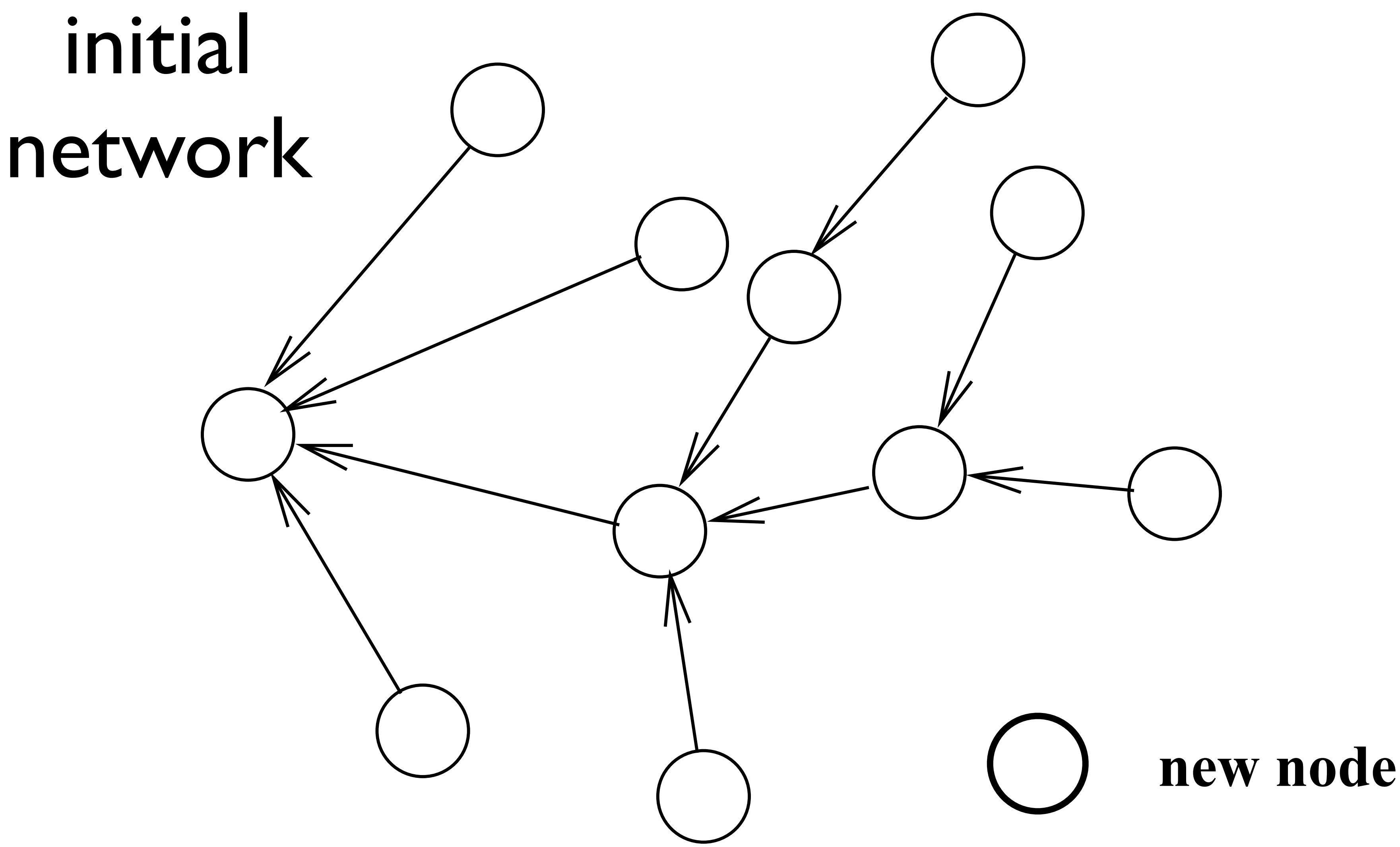

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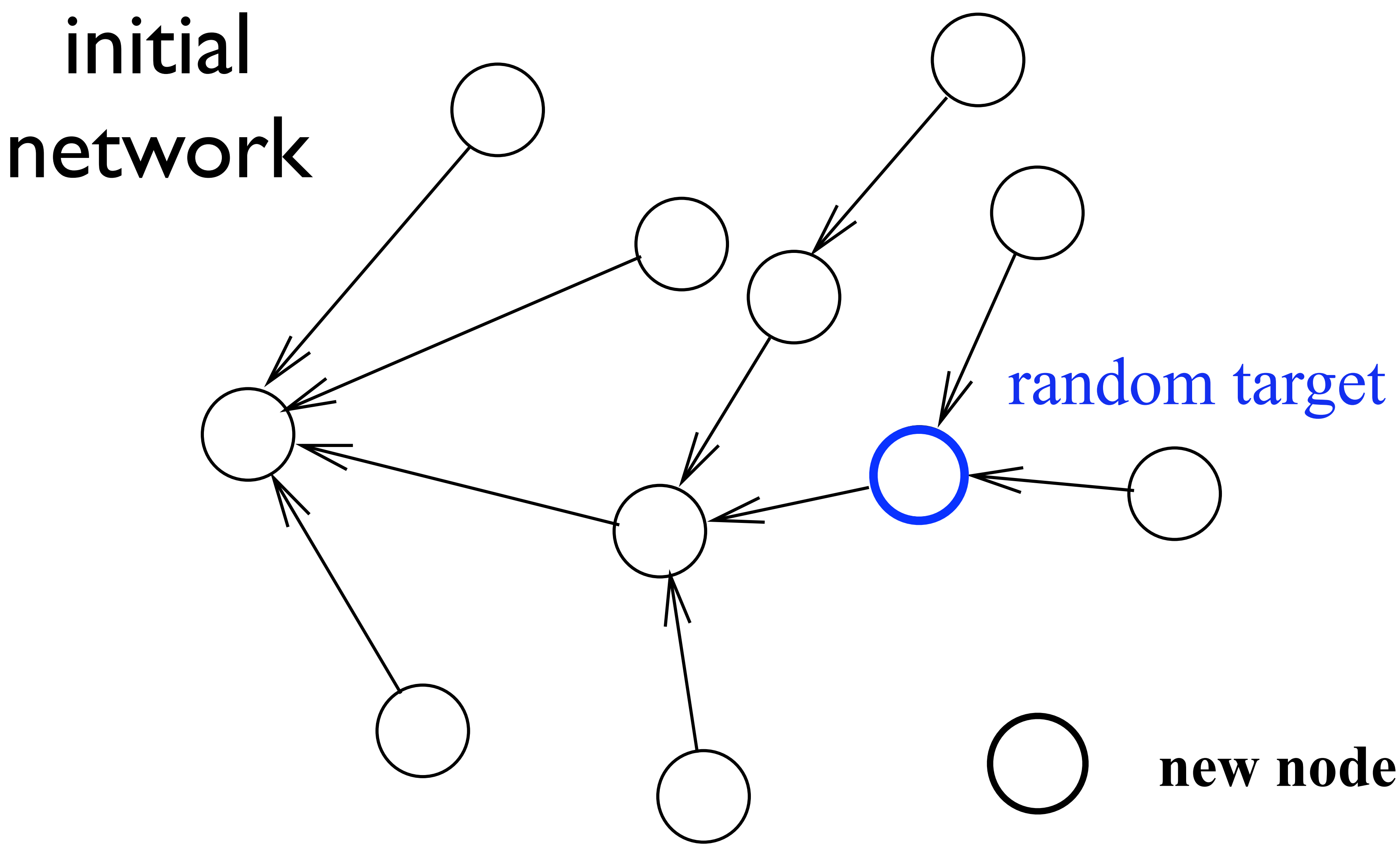

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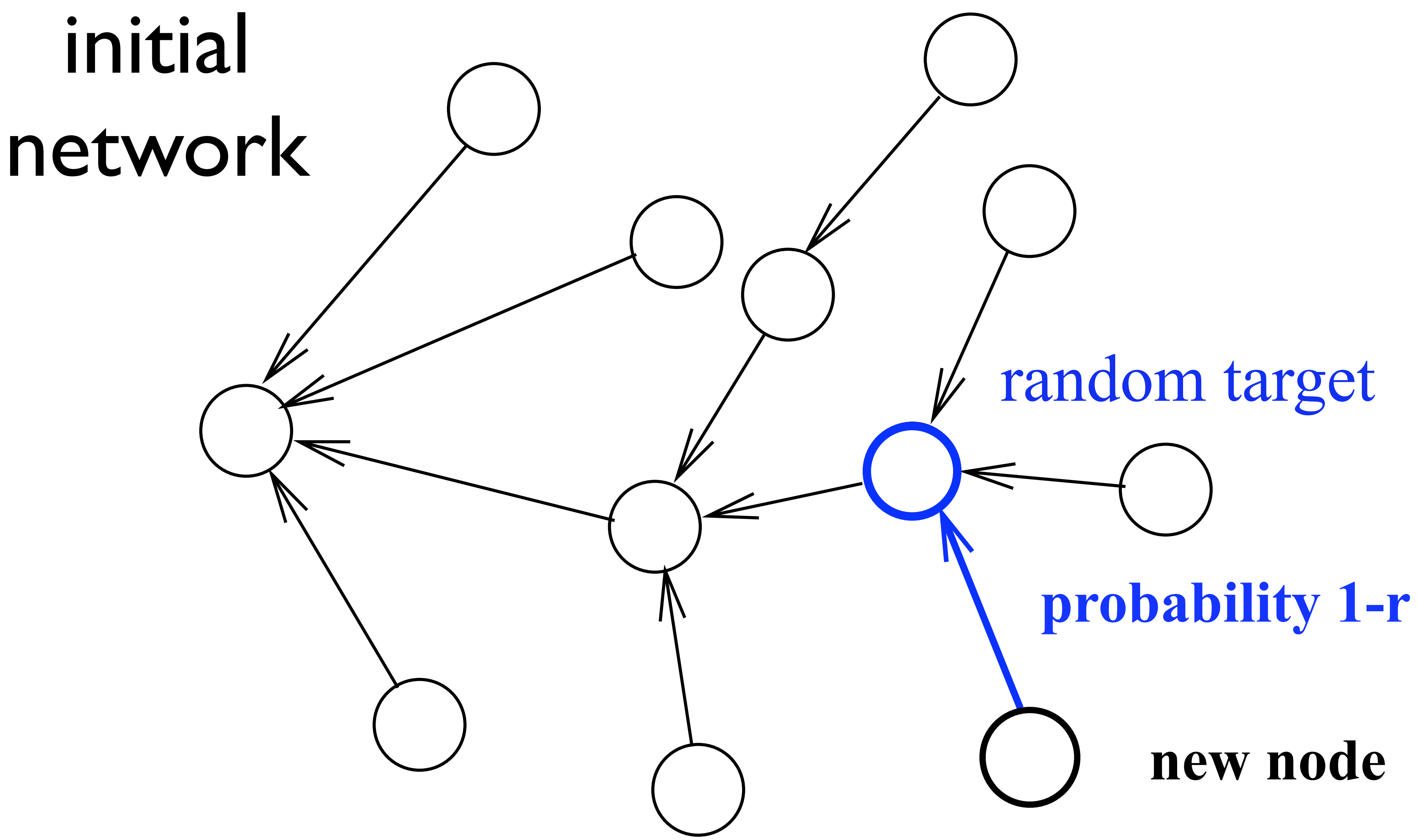

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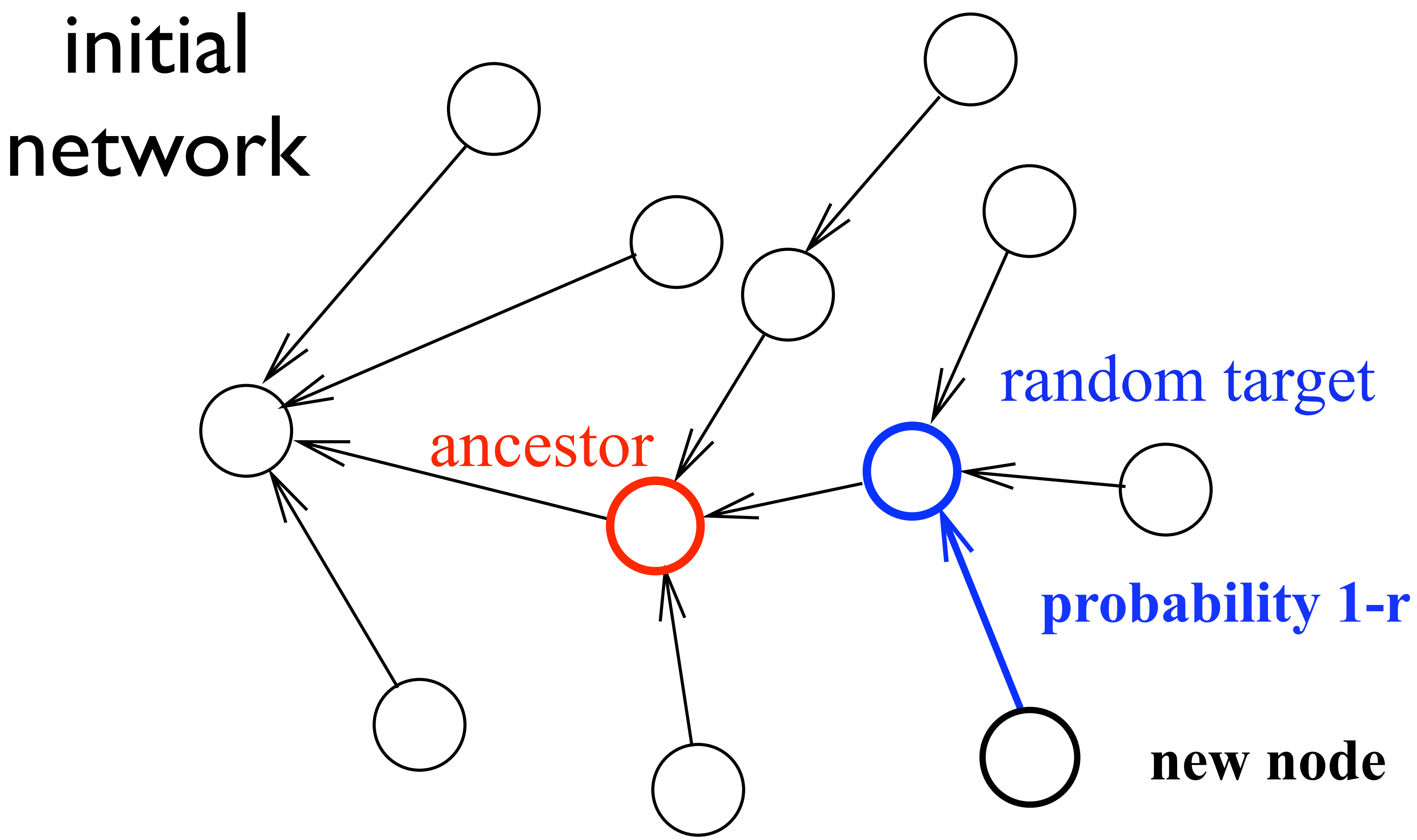

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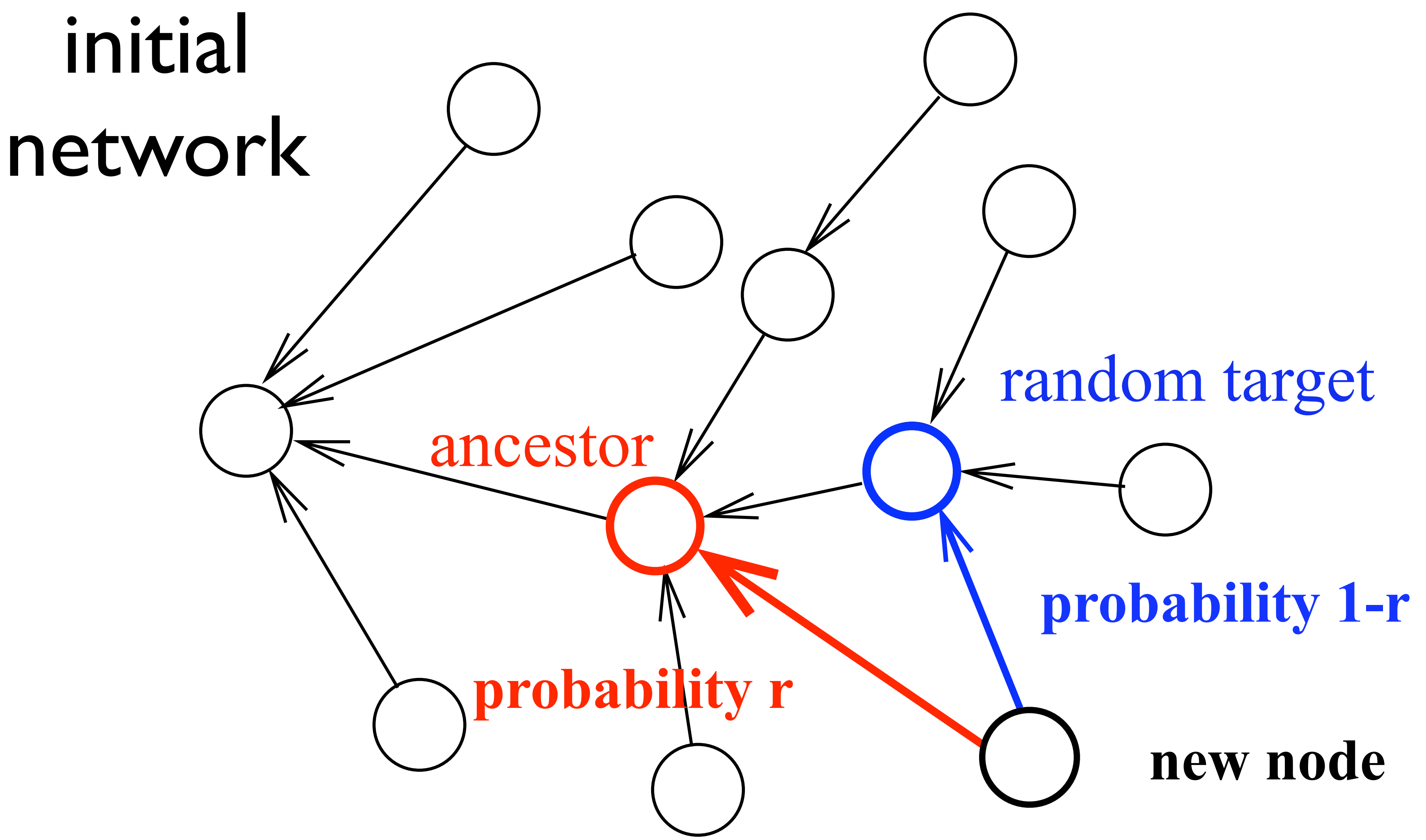

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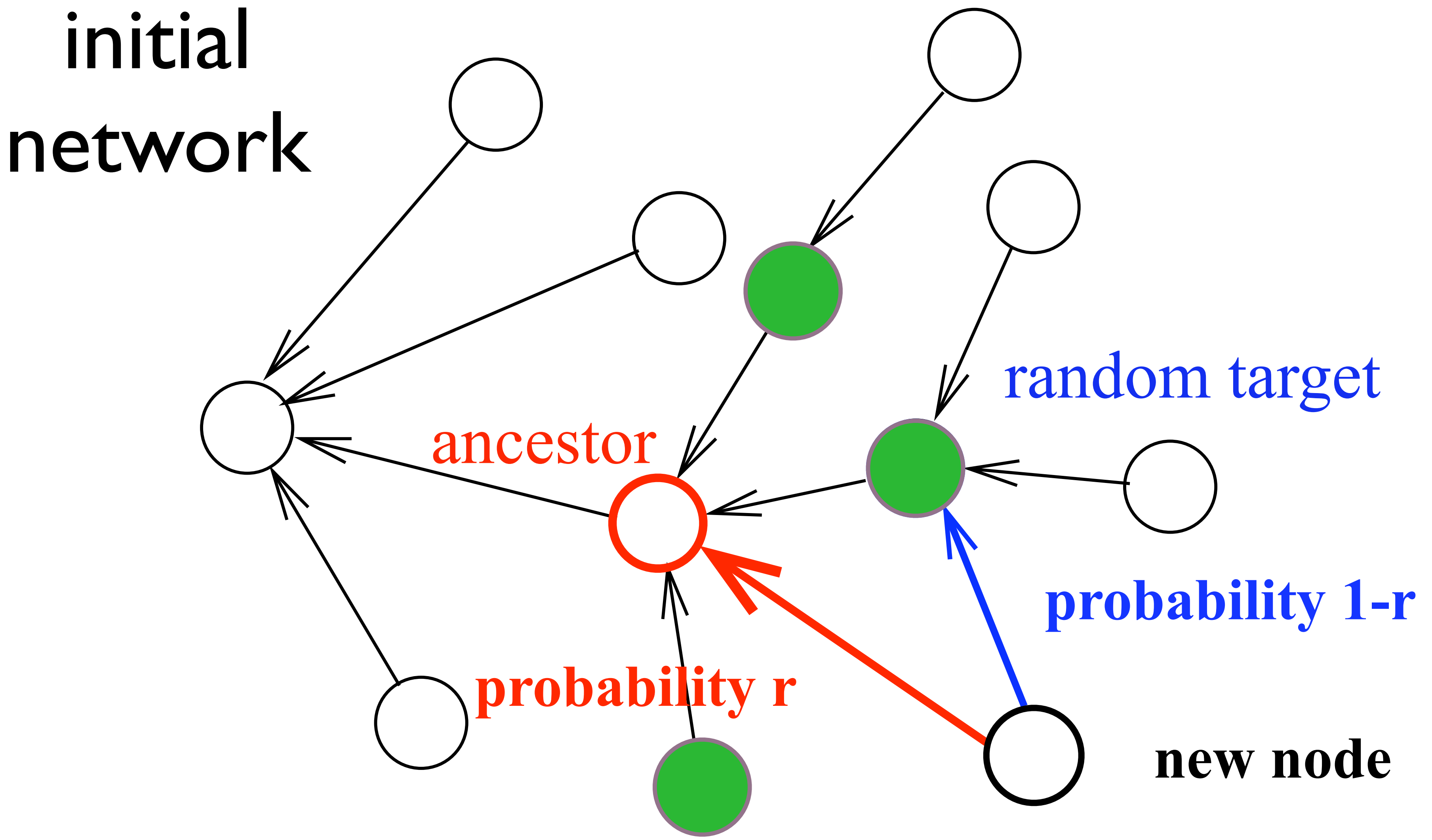
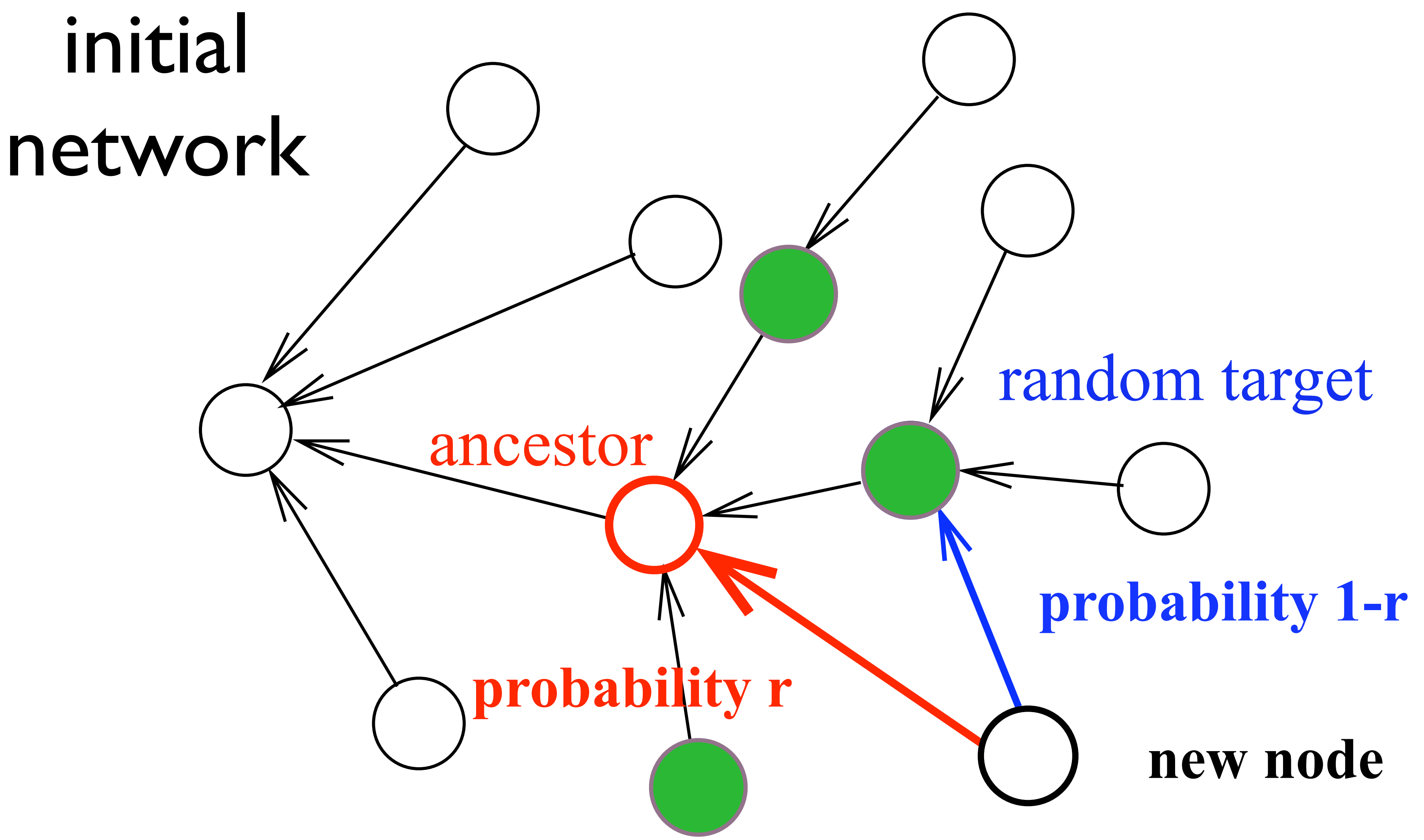


attachment rate to ancestor node:
$\propto$ number of upstream neighbors

Uniform Attachment + *Redirection*

Krapivsky & SR (2001)

= Linear* Preferential Attachment!



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Uniform Attachment + *Redirection*

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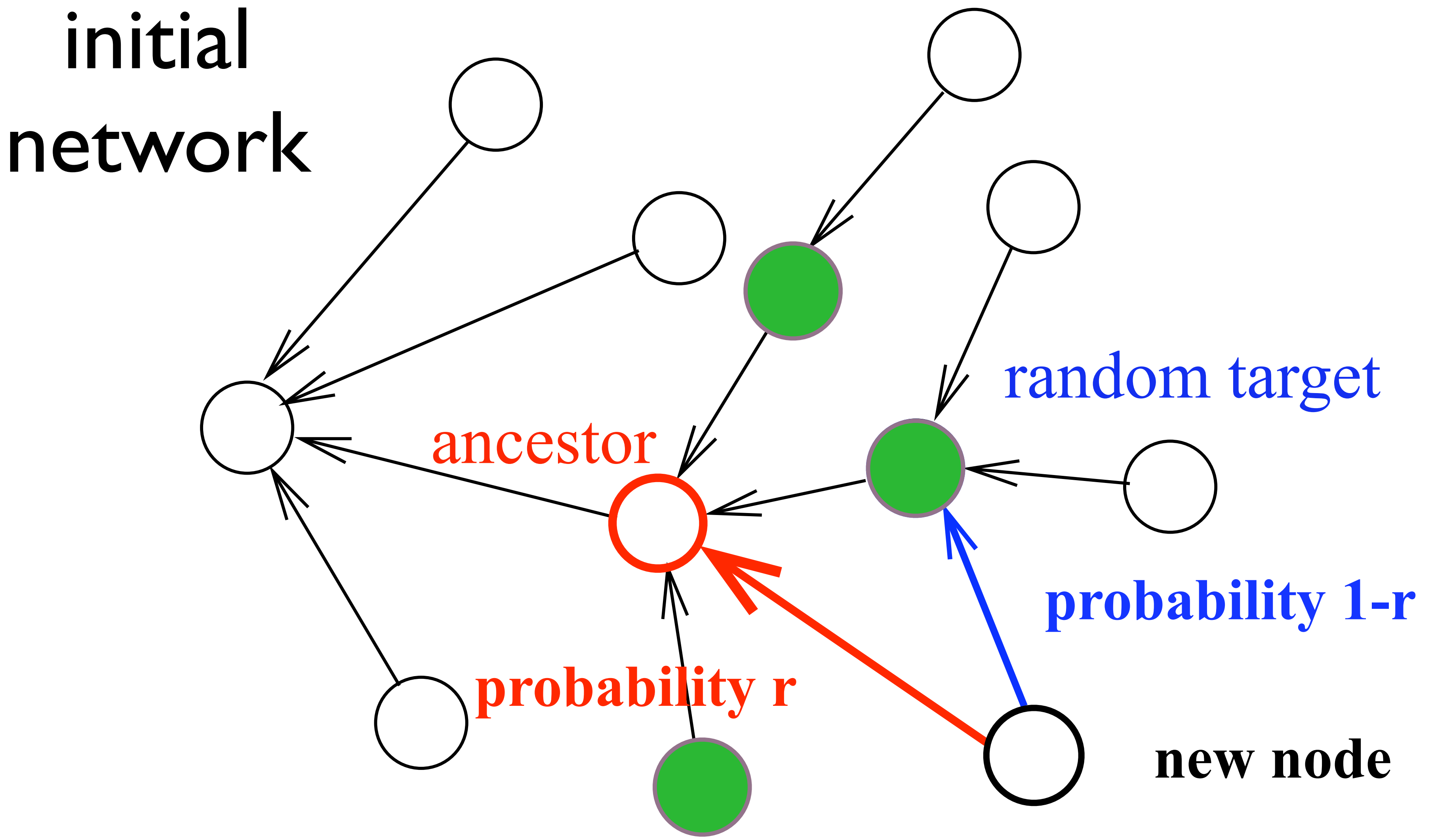
= Linear* Preferential Attachment! **shifted*



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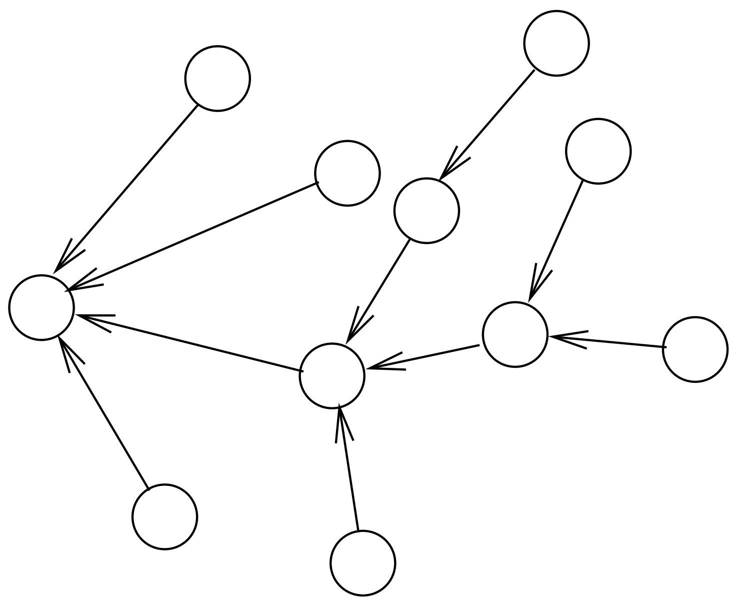

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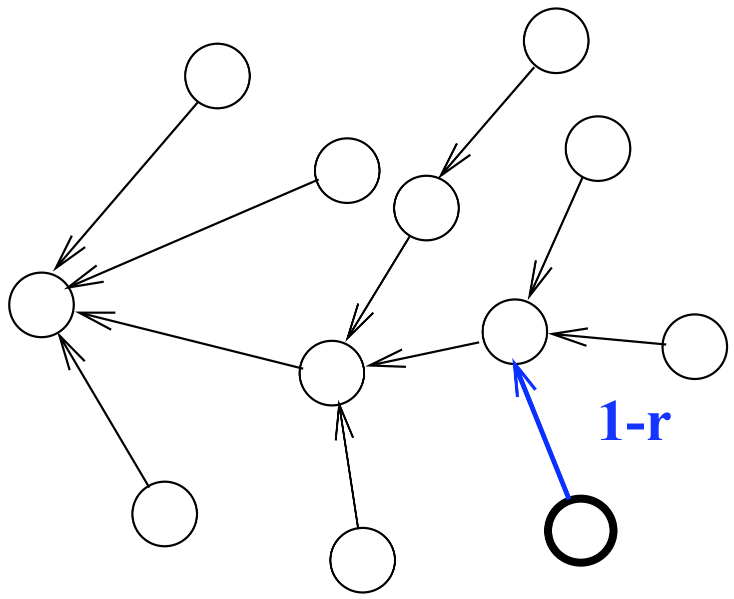
$$\frac{dN_k}{dN} = \frac{1-r}{N}\left[N_{k-1} - N_k\right] + \delta_{k,1}$$



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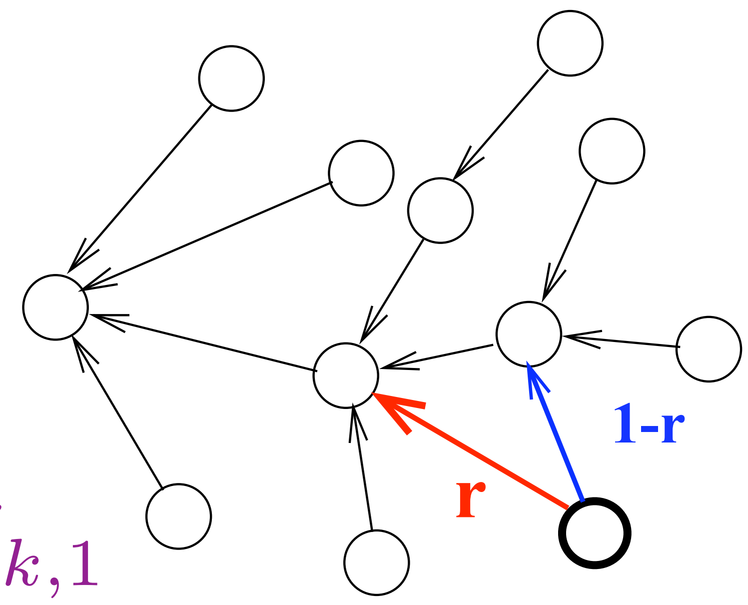

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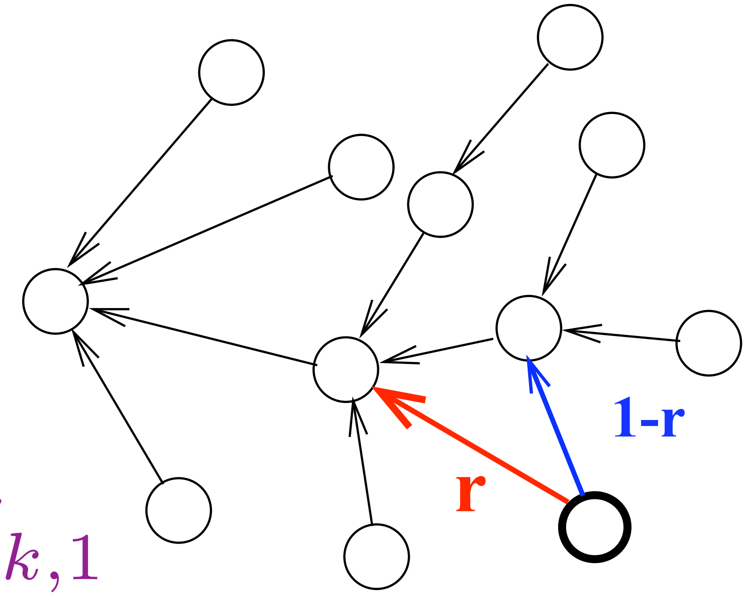

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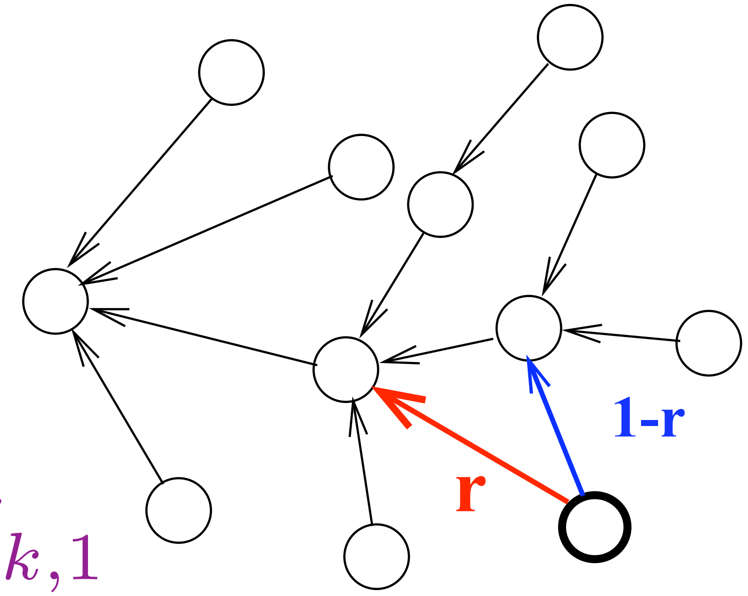

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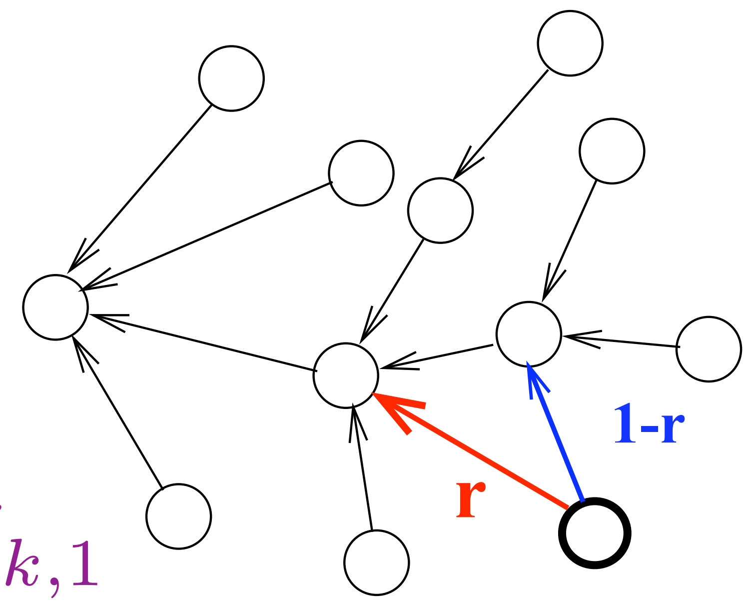

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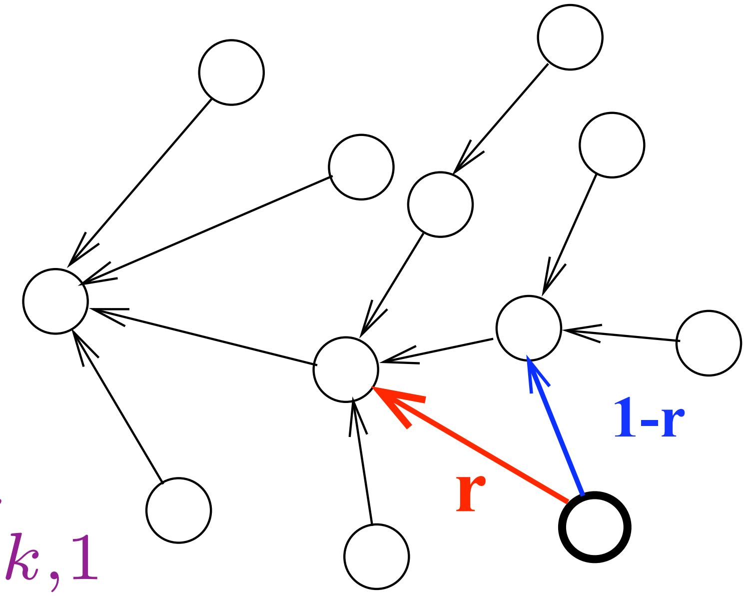
with $A_k = k + \left(\frac{1}{r} - 2\right) \equiv k + \lambda$

$A = \sum_k A_k N_k = N/r$

shifted linear attachment:

Master Equation:



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$$A = \textstyle\sum_k A_k N_k = N/r$$

shifted linear attachment:

O(1) rule → (shifted) linear preferential attachment!!

Master Equation Approach for Preferential Attachement

Krapivsky et al. (2000),
Dorogovtsev et al. (2000),
Krapivsky & SR (2001)

Master Equation:

attach to node of degree k-1 | attach to node of degree k | create node of degree 1

$$\frac{dN_k}{dN} = \frac{A_{k-1}N_{k-1}}{A} - \frac{A_k N_k}{A} + \delta_{k,1} \qquad A = \text{total rate}$$

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Degree Preferential Attachment Rate:

$$A_k \sim k^\gamma$$

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Degree Preferential Attachment Rate:

$$A_k \sim k^\gamma$$

Total Rate:

$$A = A(N) = \sum_{j=1}^{\infty} A_j N_j = \sum_{j=1}^{\infty} j^\gamma N_j \equiv M_\gamma(N)$$

Moment equations:

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These suggest: $A(N) = \sum_j j^\gamma N_j \propto \mu(\gamma)N \quad \text{for } 0 \leq \gamma \leq 1$

$$N_k(N) \equiv N\, n_k$$

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Converts rate eqns. to linear recursions

$$\frac{dN_k}{dN} = \frac{A_{k-1}N_{k-1}}{A} - \frac{A_k N_k}{A} + \delta_{k,1}$$

$$\Rightarrow \quad n_k = \frac{A_{k-1}n_{k-1} - A_k n_k}{\mu} + \delta_{k,1}$$

Formal Solution

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$$n_k = \frac{A_{k-1}n_{k-1} - A_k n_k}{\mu} + \delta_{k,1} \longrightarrow n_k = \frac{\mu}{A_k}\prod_{j=1}^{k}\left(1 + \frac{\mu}{A_k}\right)^{-1}$$

Formal Solution

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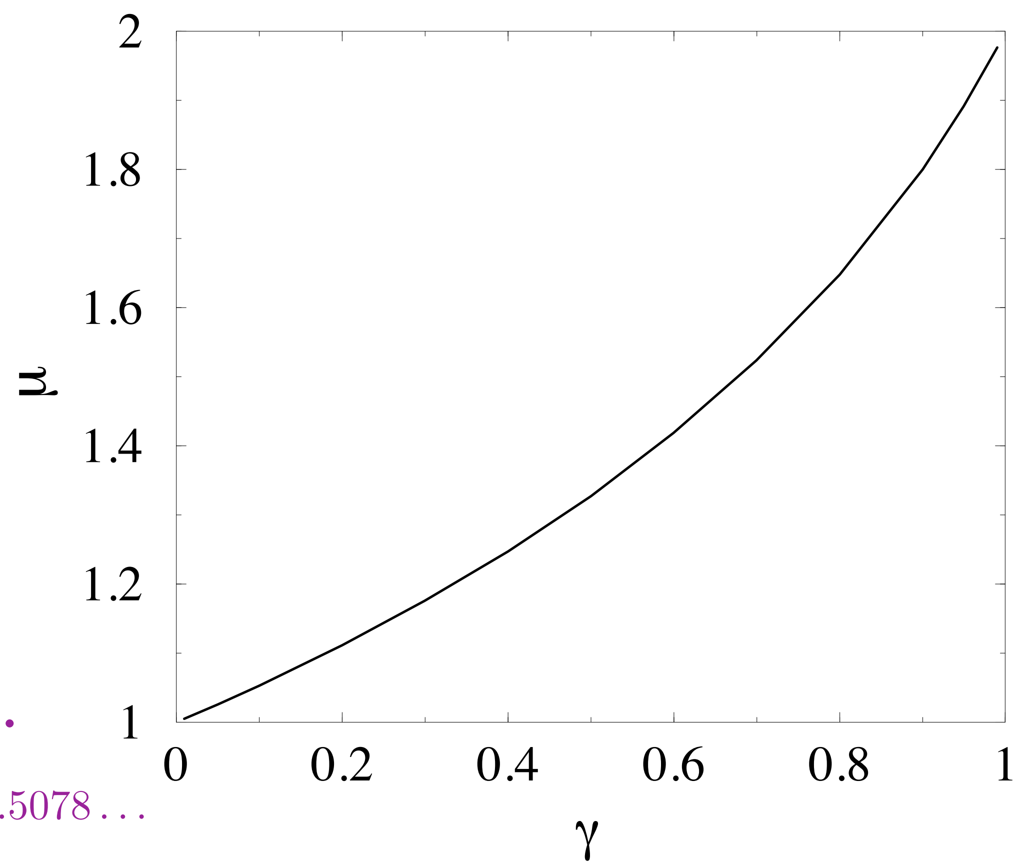
$$\text{determination of } \mu = \sum_{k=1}^{\infty} A_k n_k: \quad \longrightarrow 1 = \sum_{k=1}^{\infty}\prod_{j=1}^{k}\left(1 + \frac{\mu}{A_k}\right)^{-1}$$

numerical solution for $A_k = k^\gamma$



$$\mu = 2 - C(1-\gamma) + \ldots$$

$$C = 4\sum_{j=1}^{\infty}\frac{\ln j}{(j+1)(j+2)} = 2.407\ldots$$

$$\mu = 1 + B\gamma + \ldots$$

$$B = \sum_{j=1}^{\infty}\frac{\ln j}{2^j} = 0.5078\ldots$$

Formal Solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$

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Asymptotics for $A_k \sim k^\gamma$

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Asymptotics for $A_k \sim k^\gamma$ **(use** $\prod_j^k (1+\epsilon_j)^{-1} \sim e^{-\int^k \epsilon_j \, dj}$ **)**

$$n_k \sim \begin{cases} k^{-\gamma}\, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \le \gamma < 1 \\ & \\ & \end{cases}$$

Formal Solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$

Asymptotics for $A_k \sim k^{\gamma}$ (use $\prod_{j}^{k} (1+\epsilon_j)^{-1} \sim e^{-\int^k \epsilon_j \, dj}$)

$$n_k \sim \begin{cases} k^{-\gamma} \, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \le \gamma < 1 \\ & \\ & \end{cases}$$

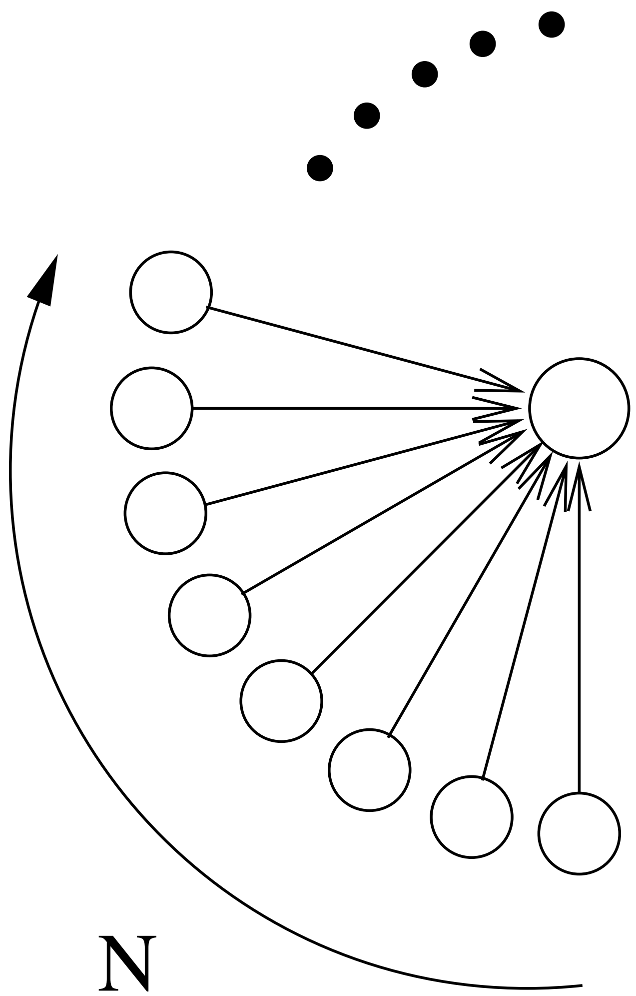
universal, generic

Formal Solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$

Asymptotics for $A_k \sim k^\gamma$

$$n_k \sim \begin{cases} k^{-\gamma}\, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \le \gamma < 1 \\ \\ \\ \text{“bible”} & \gamma > 2 \end{cases}$$

Creating a "Bible"



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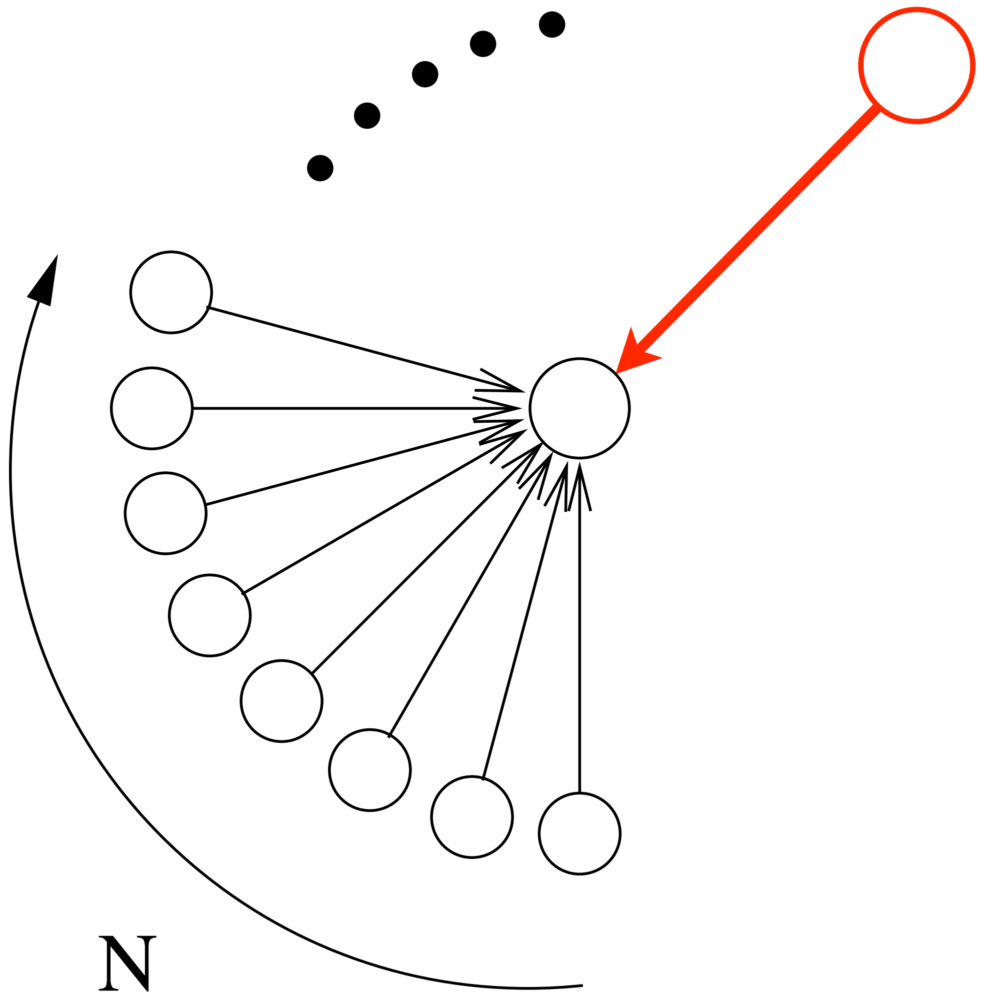

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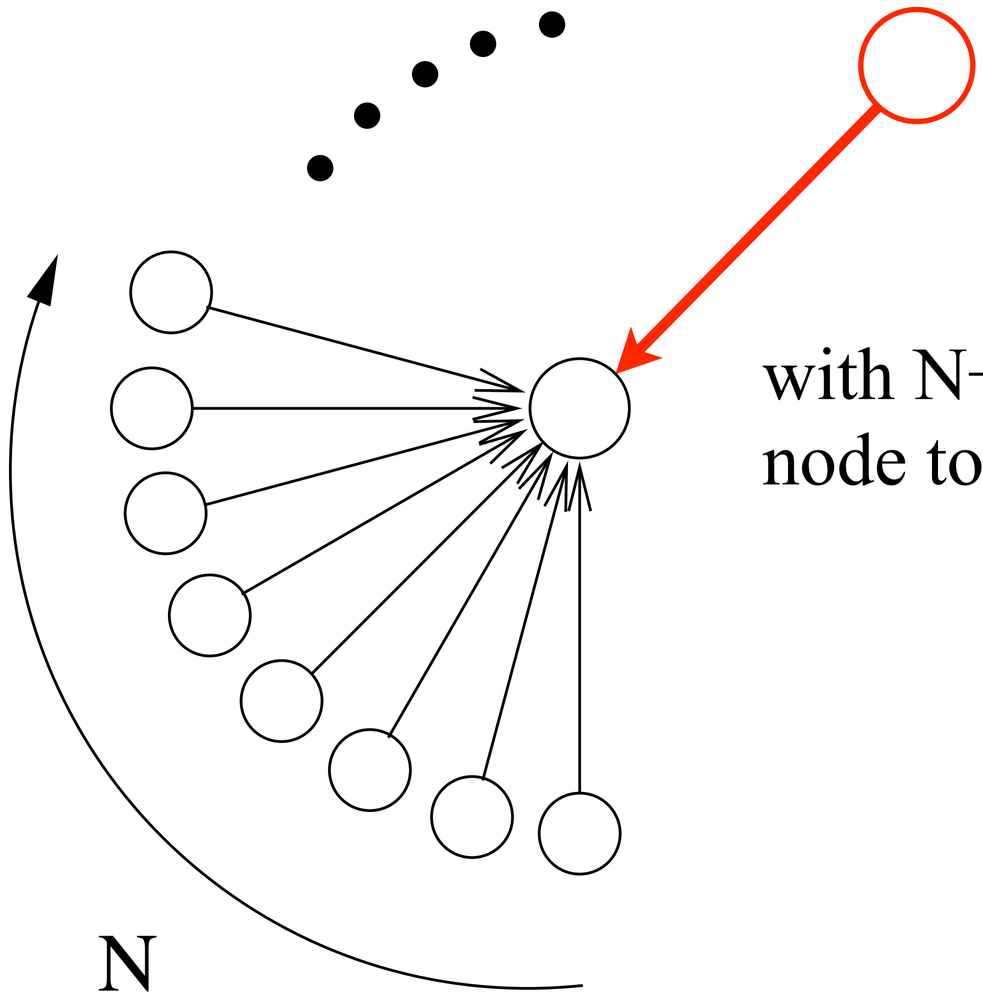

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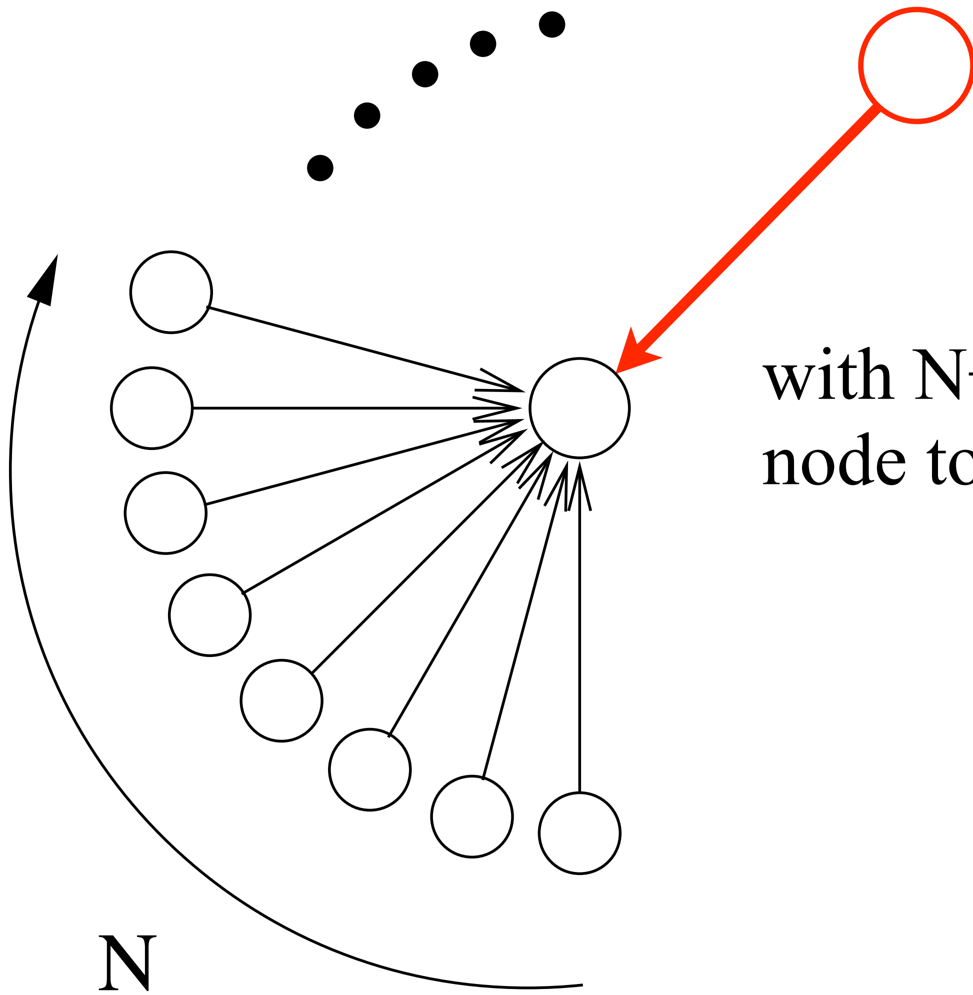

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bible probability: $$\mathcal{P} = \prod_{N=1}^{\infty} \frac{1}{1+N^{1-\gamma}} = \begin{cases} \text{zero} & \gamma \leq 2 \\ \text{non-zero} & \gamma > 2 \end{cases}$$

Formal Solution: $n_k = \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1}$

Asymptotics for $A_k \sim k^\gamma$

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$$
n_k \sim \begin{cases}
k^{-\gamma}\, e^{-[\mu k^{1-\gamma}/(1-\gamma)]} & 0 \leq \gamma < 1 \\[2ex]
k^{-\nu},\ \nu > 2 \text{ !!!} & \gamma = 1 \\[2ex]
\text{“best seller”} & 1 < \gamma \leq 2 \\[2ex]
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\end{cases}
$$

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\begin{aligned}
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$$\longrightarrow 1 < \mu < \infty$$

$$2 < \nu < \infty$$

Linear Attachment $A_k = k$:

$$n_k = \frac{4}{k(k+1)(k+2)}$$

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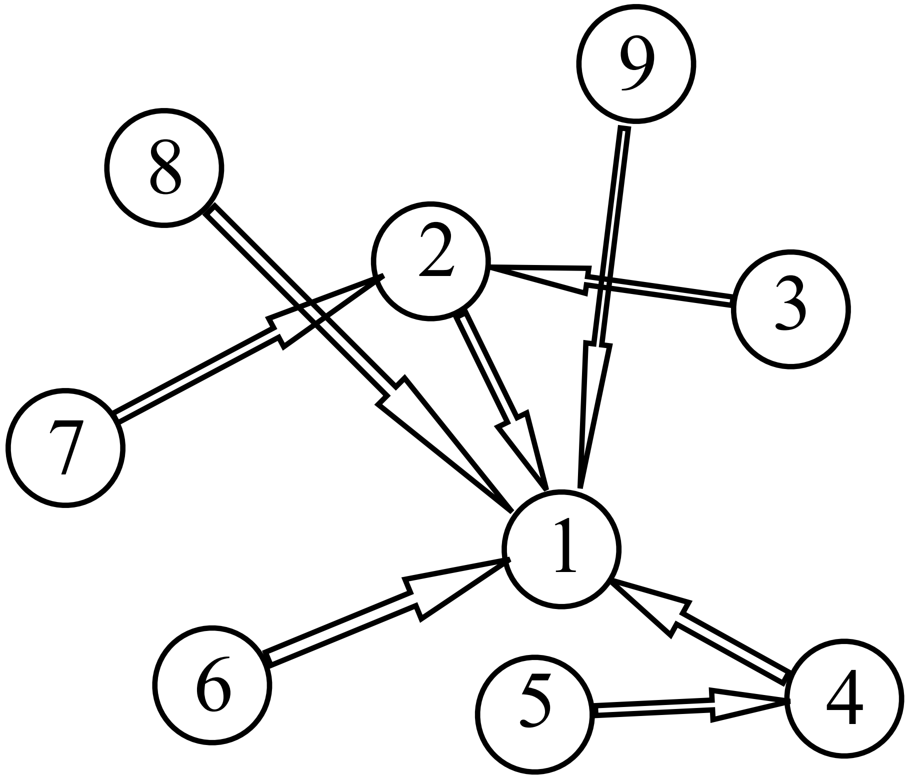
Shifted Linear Attachment $A_k = k + \lambda$

$$\begin{aligned} n_k &= \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1} \\ &= (2+\lambda)\,\frac{\Gamma(3+2\lambda)}{\Gamma(1+\lambda)}\,\frac{\Gamma(k+\lambda)}{\Gamma(k+3+2\lambda)} \\ &\sim k^{-(3+\lambda)} \qquad -1 < \lambda < \infty \end{aligned}$$

Genealogy (for uniform attachment)



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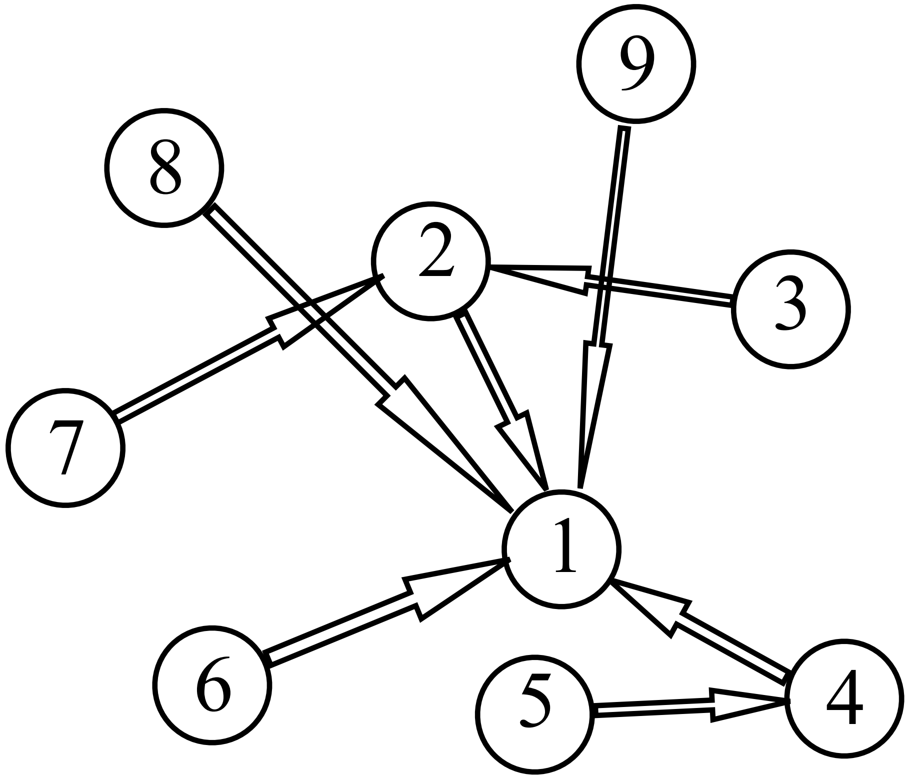
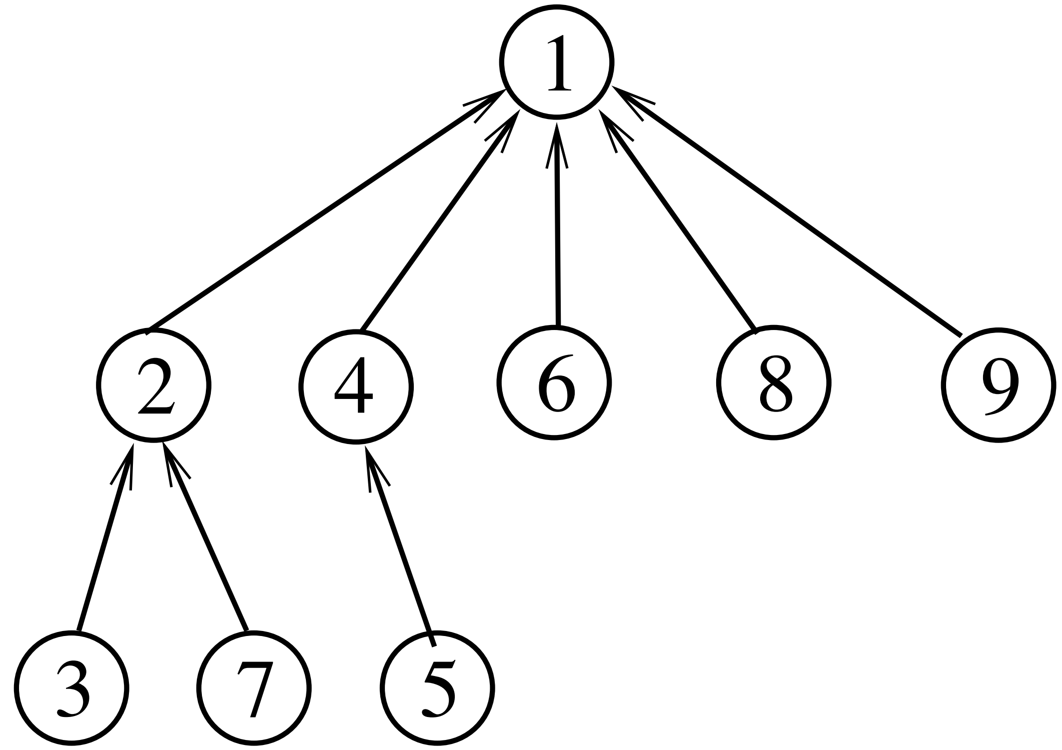


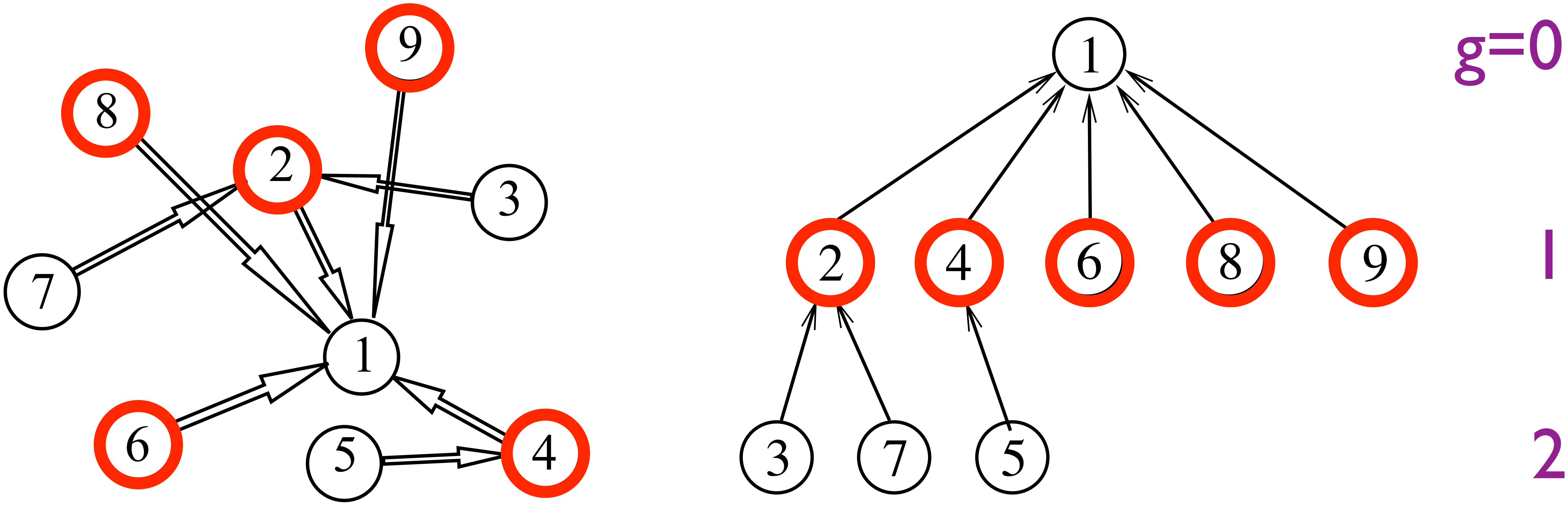


g=0

1

2

Genealogy (for uniform attachment)



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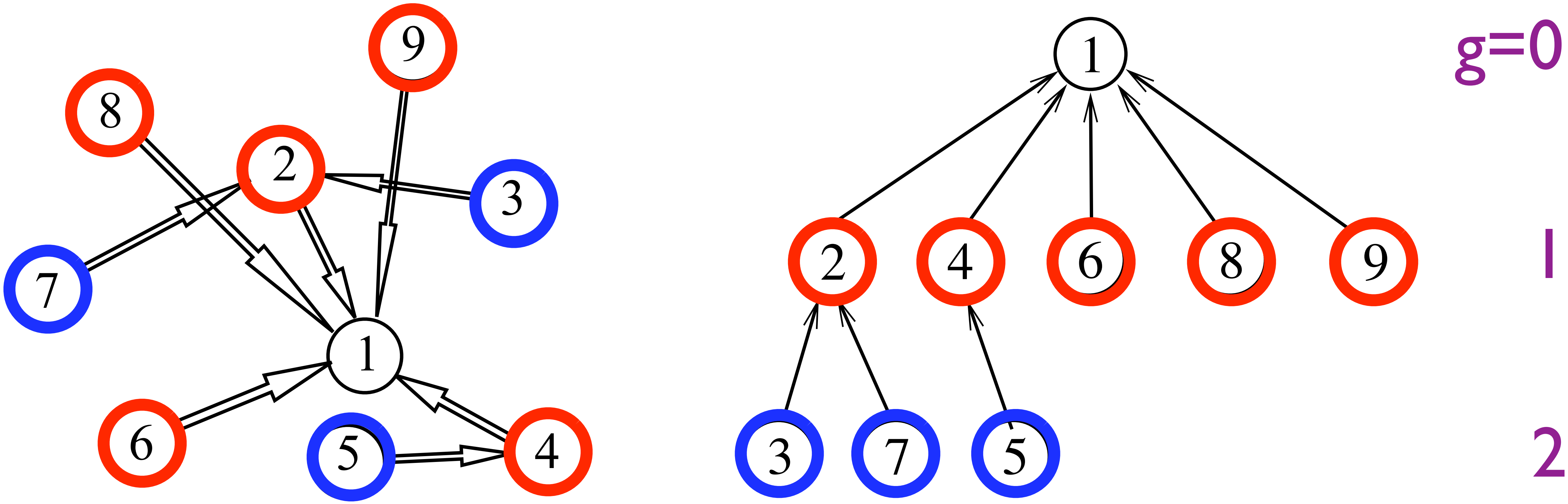

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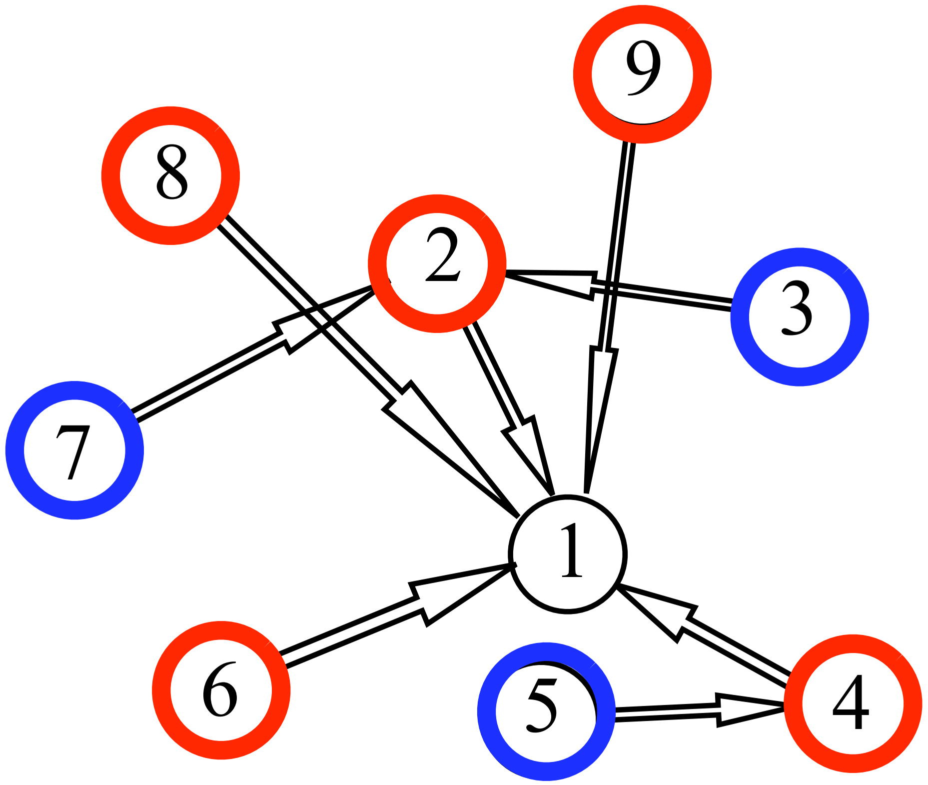

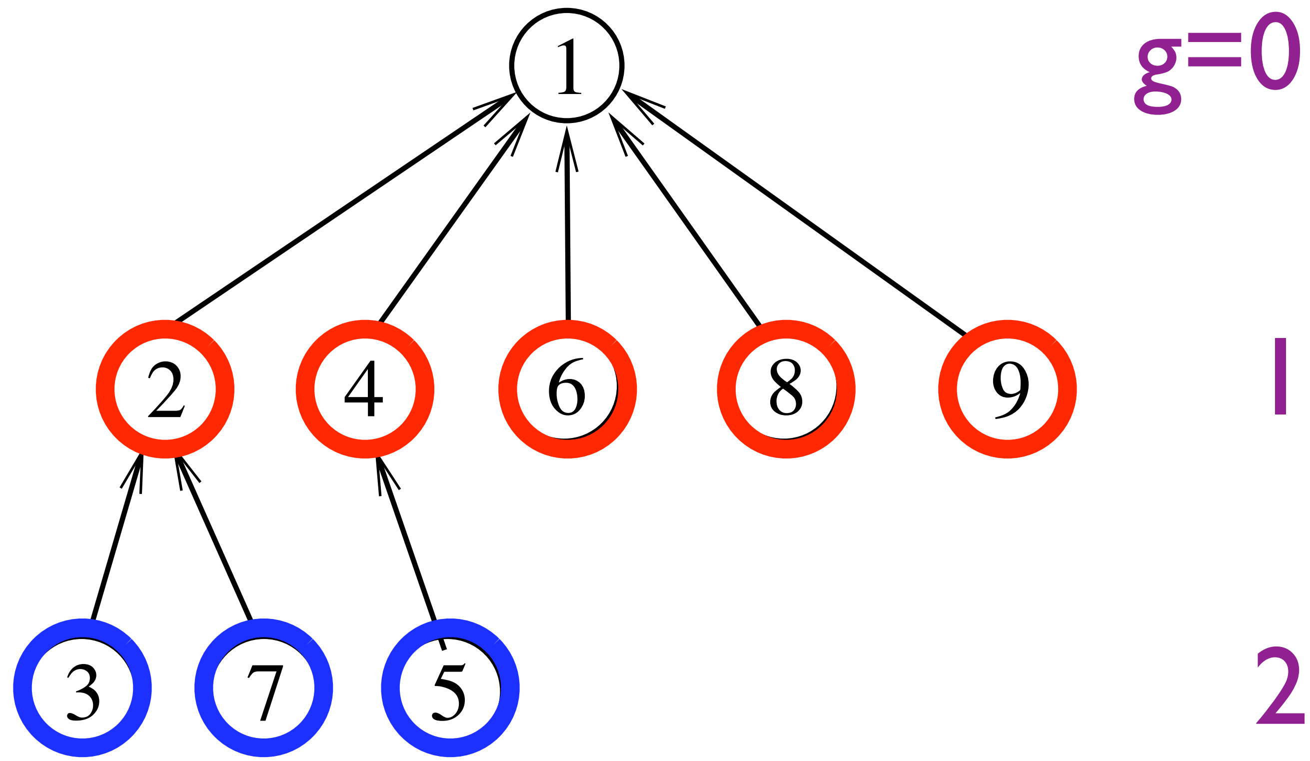

Size of generation g ≡ L_g

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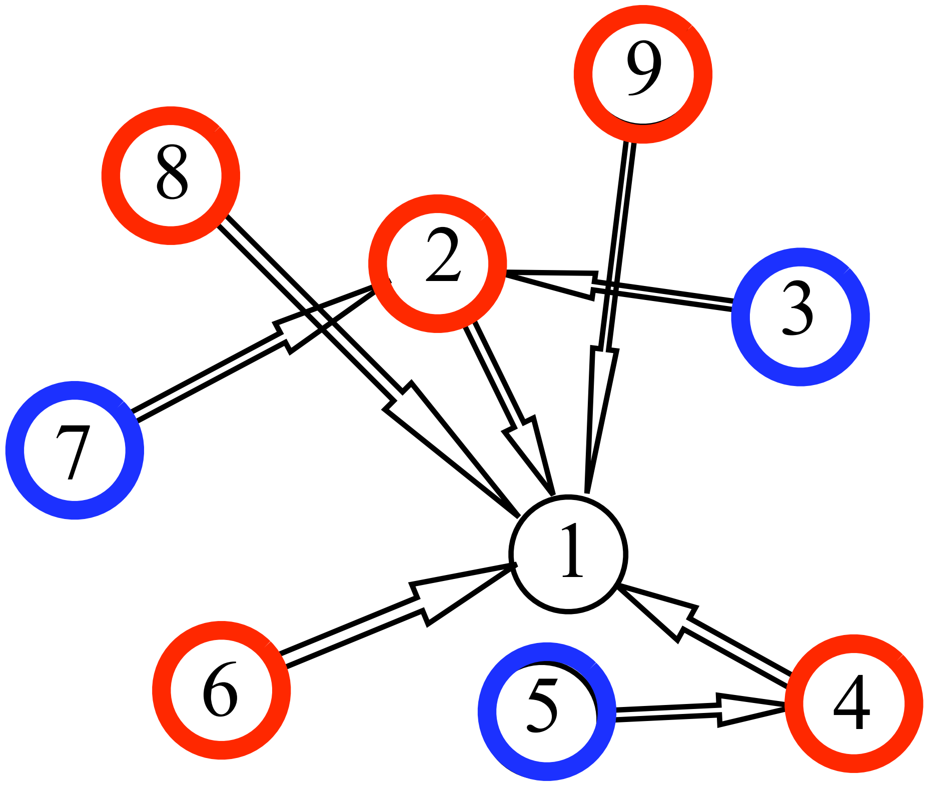

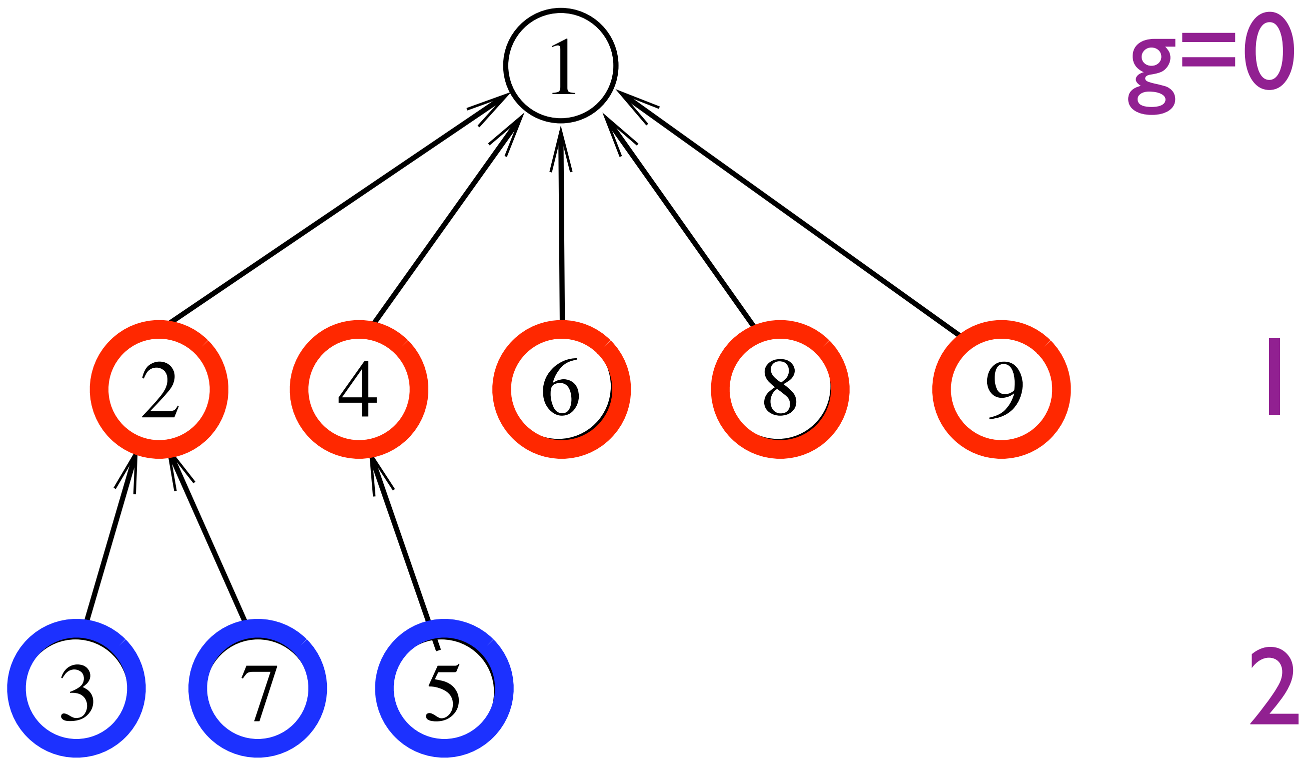

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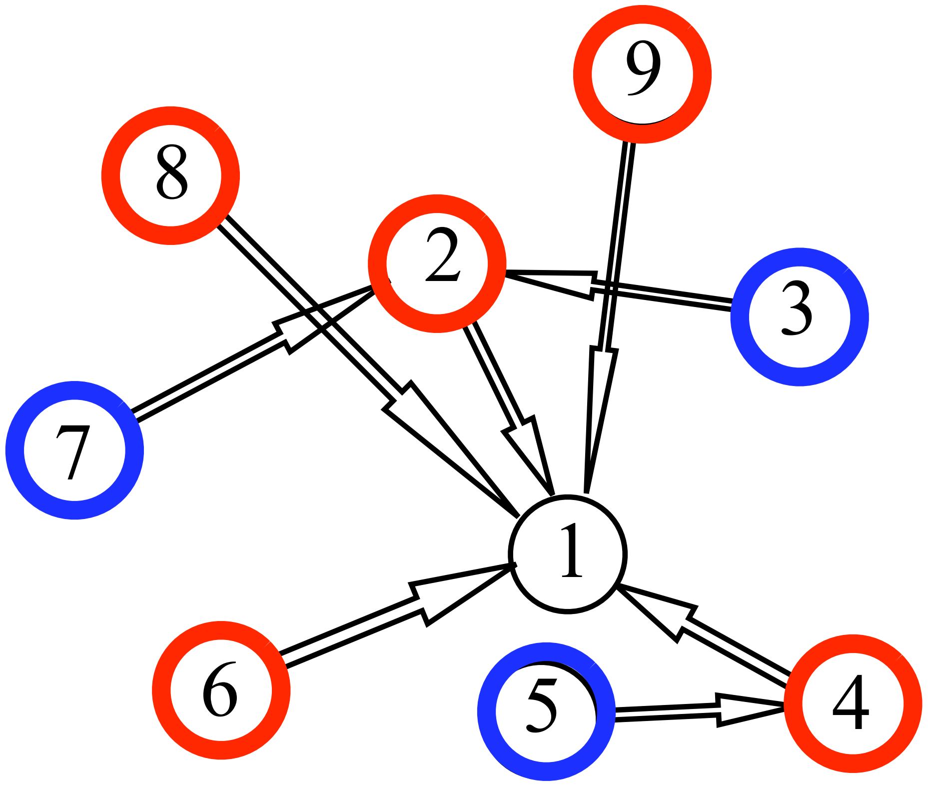

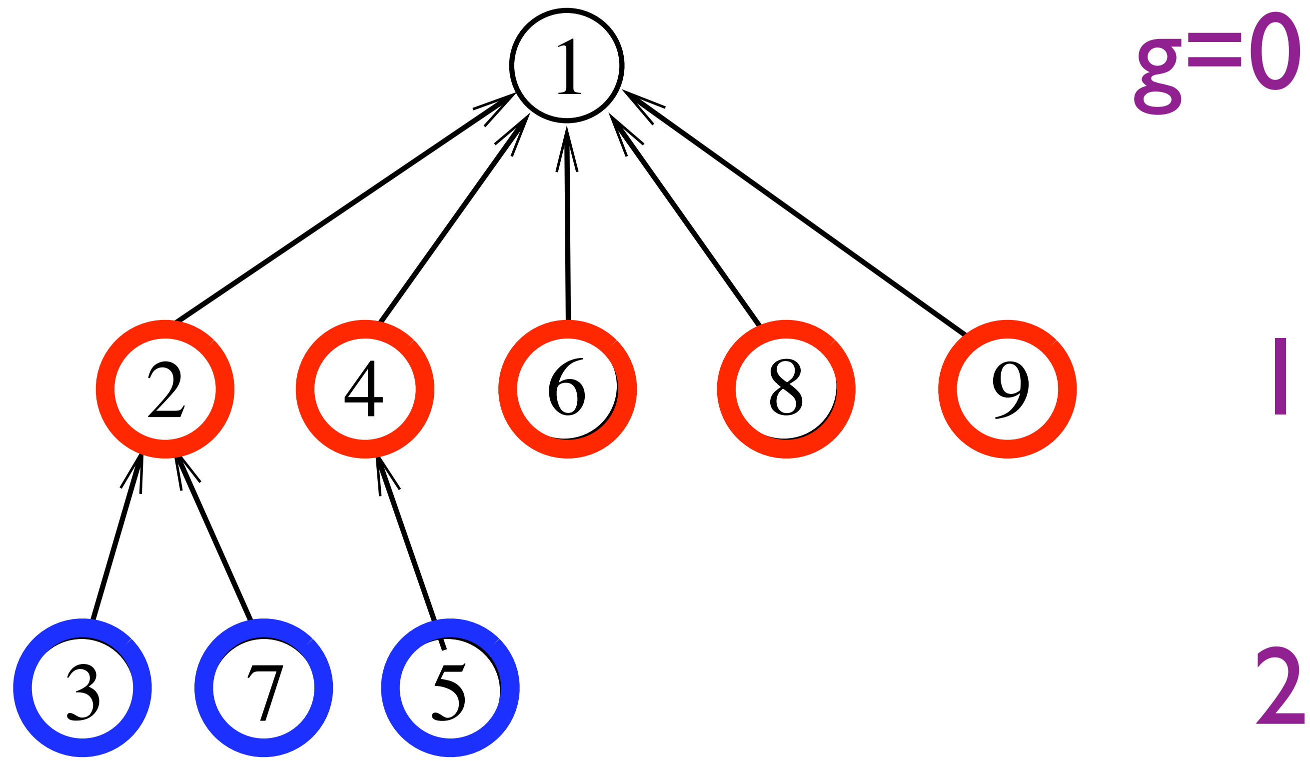

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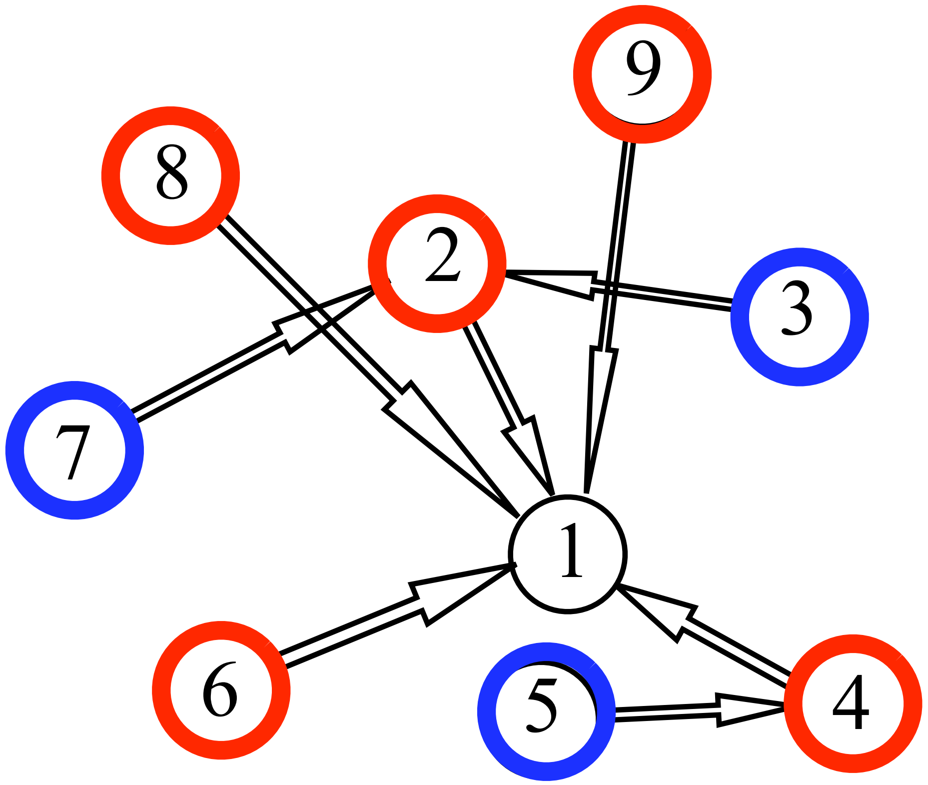

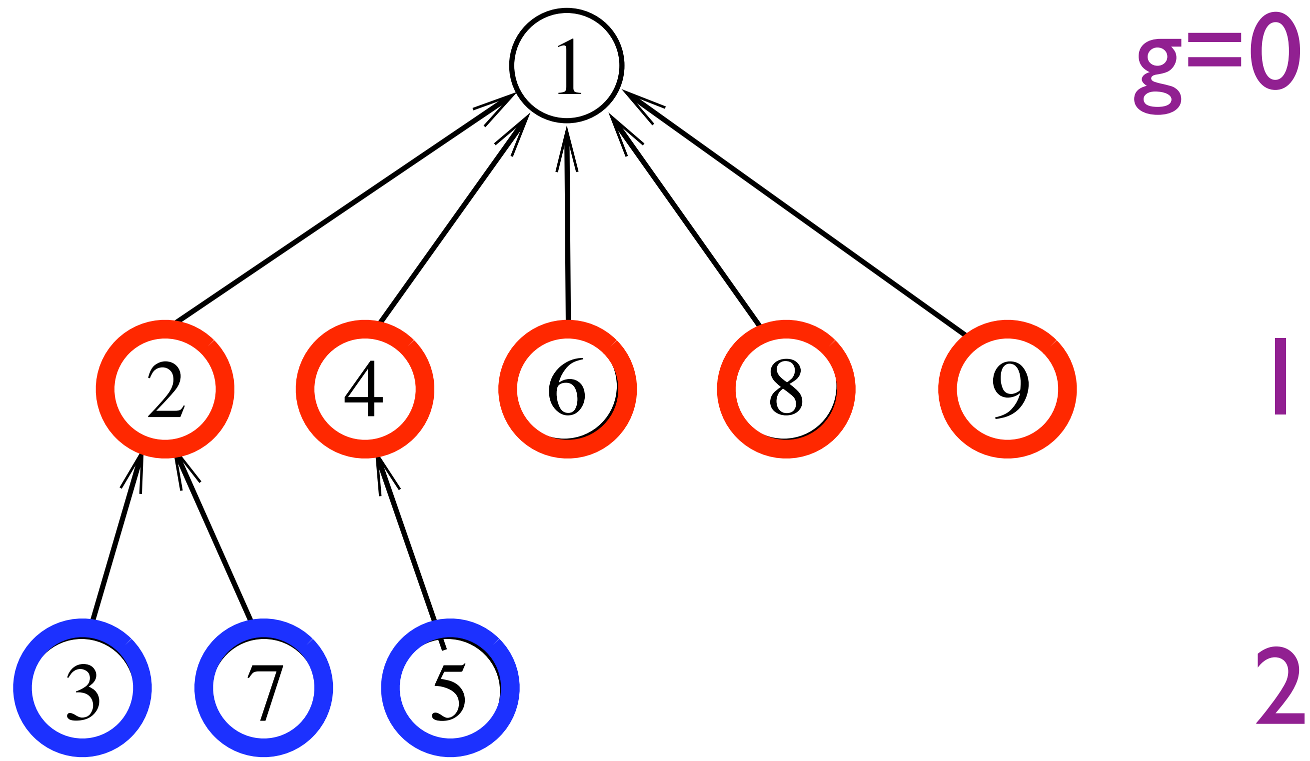

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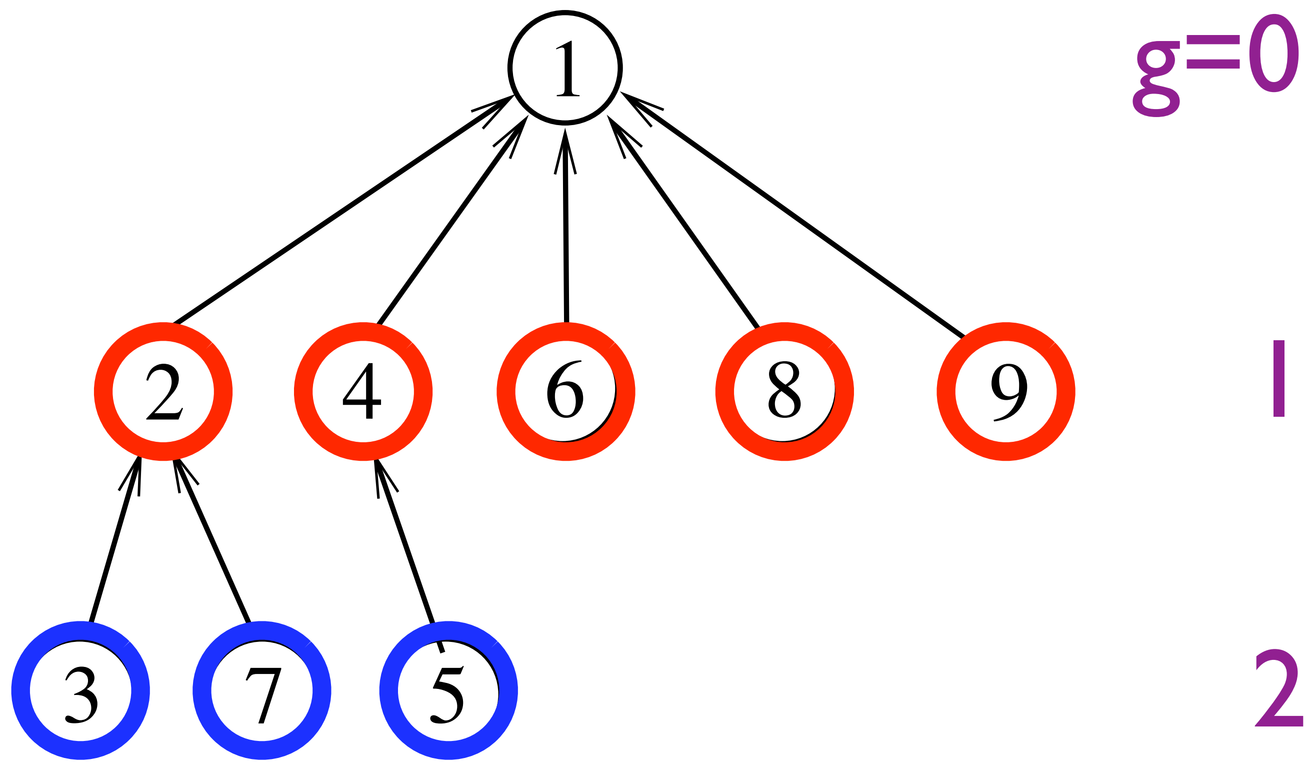
$L_g = 1$ defines last generation $\longrightarrow g_{\max} \sim e \ln N$

$\text{diameter} \sim 2e \ln N$

Genealogy (for A_k=k) *use redirection!*



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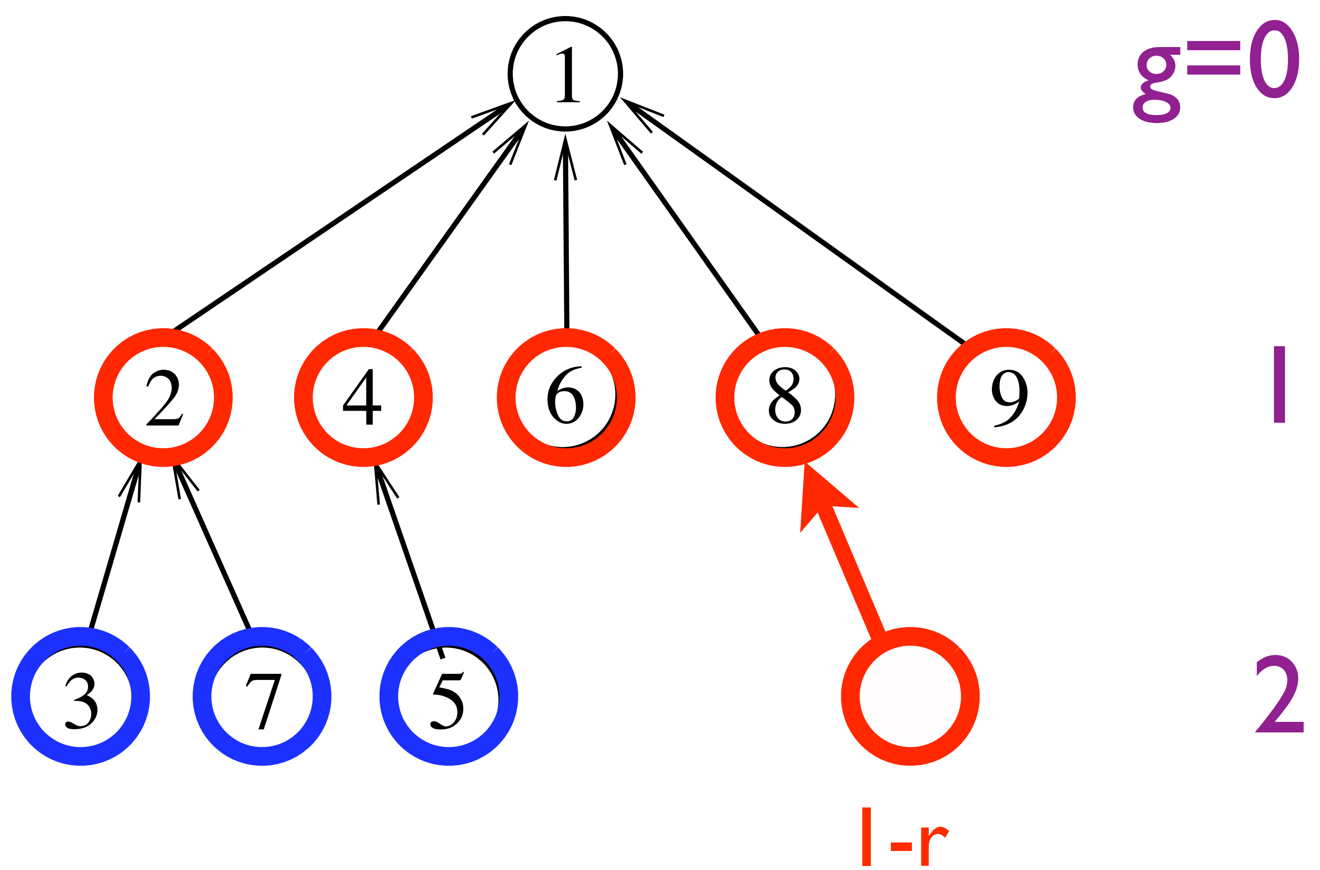

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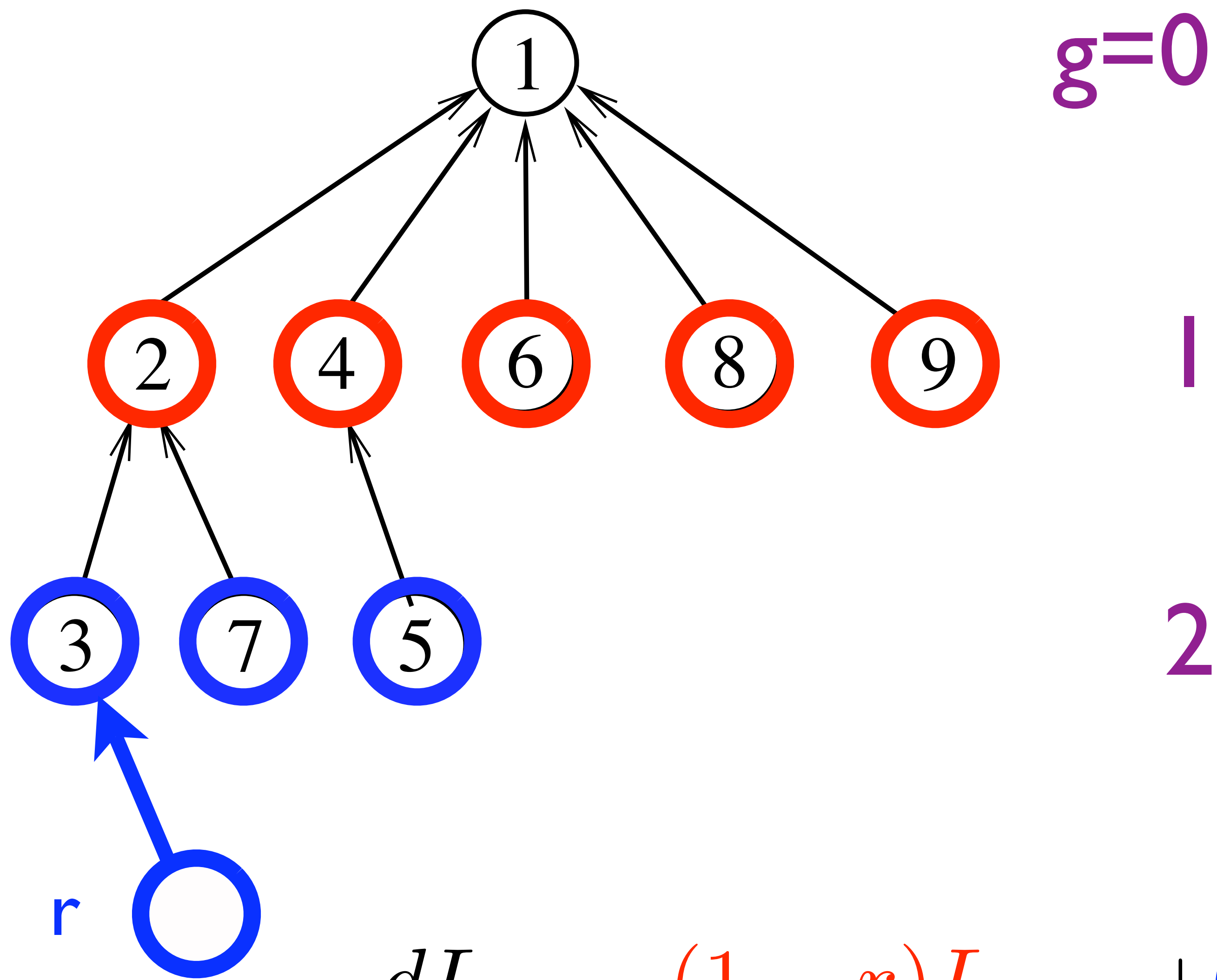

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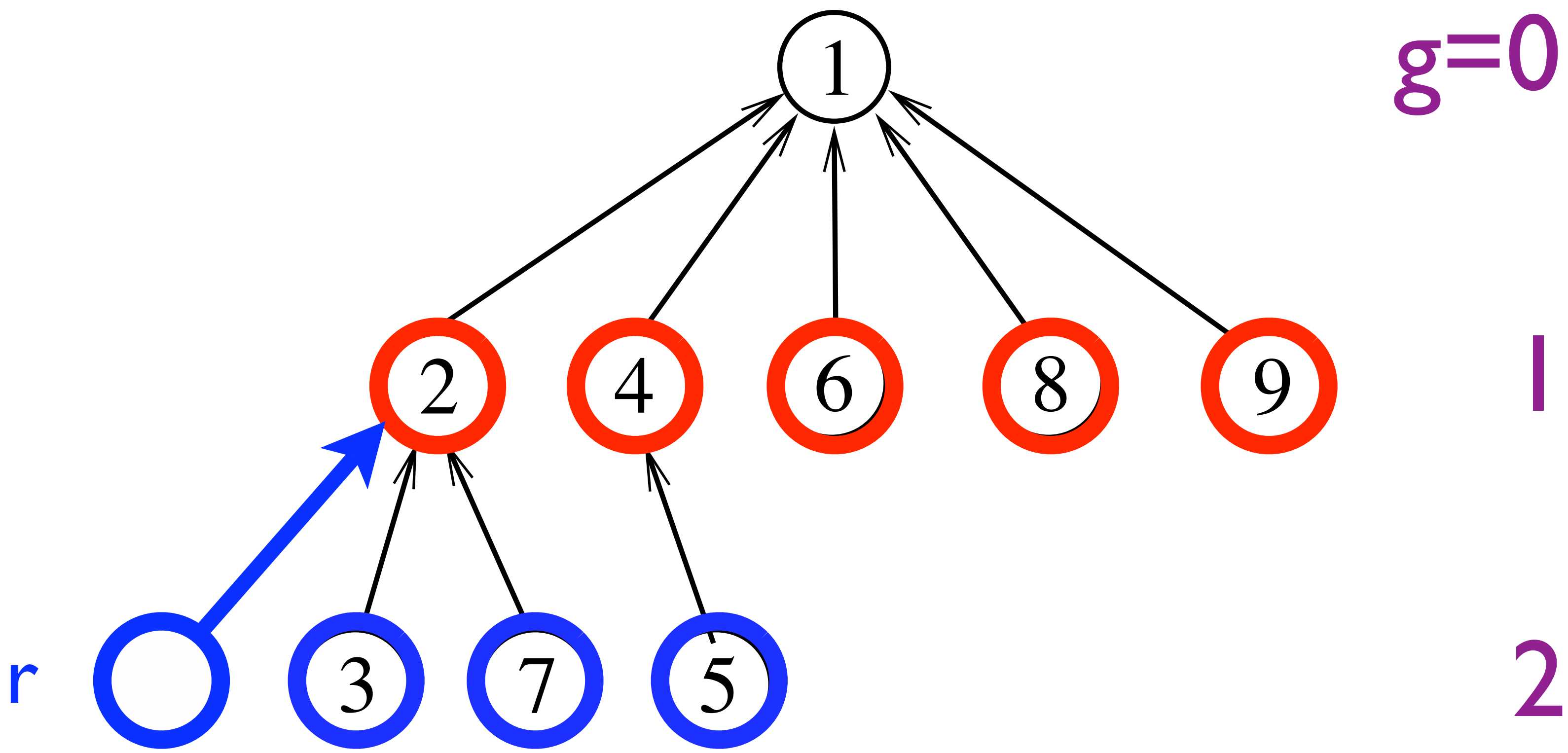

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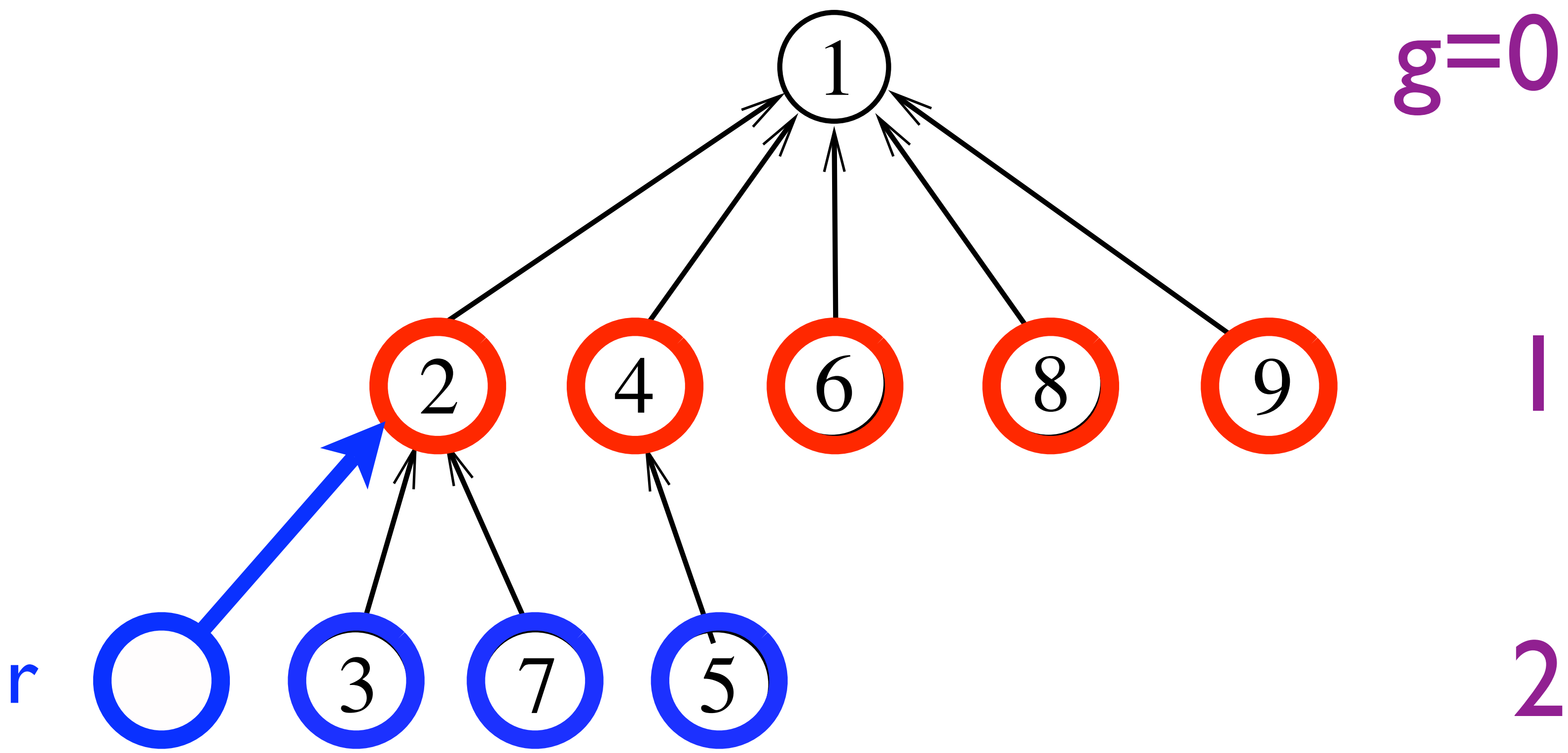

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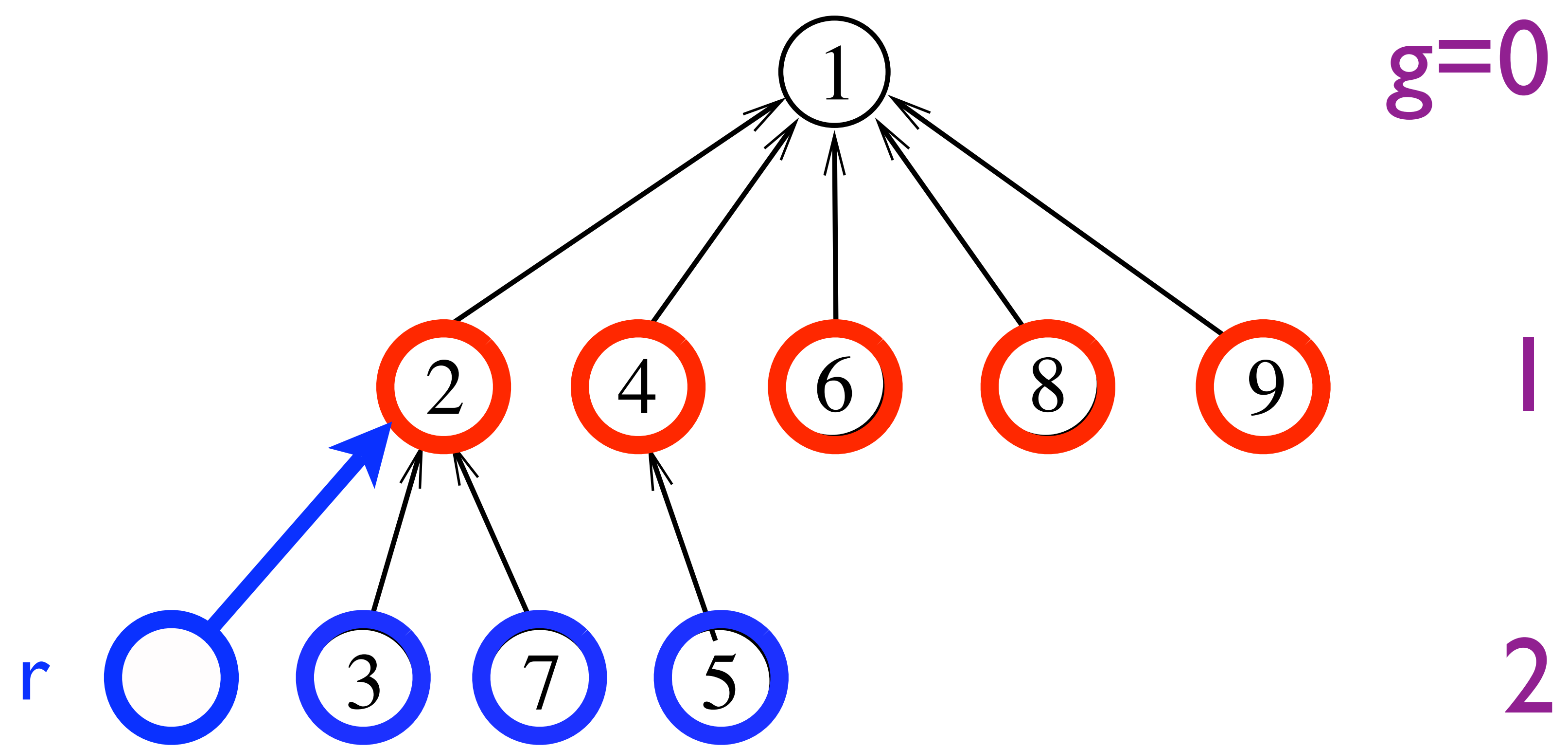

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Outlook

The master equation is a powerful tool to analyze incrementally growing, complex networks.

Wide range of degree distributions arise by preferential attachment; *non-universal power law for linear preferential attachment.*

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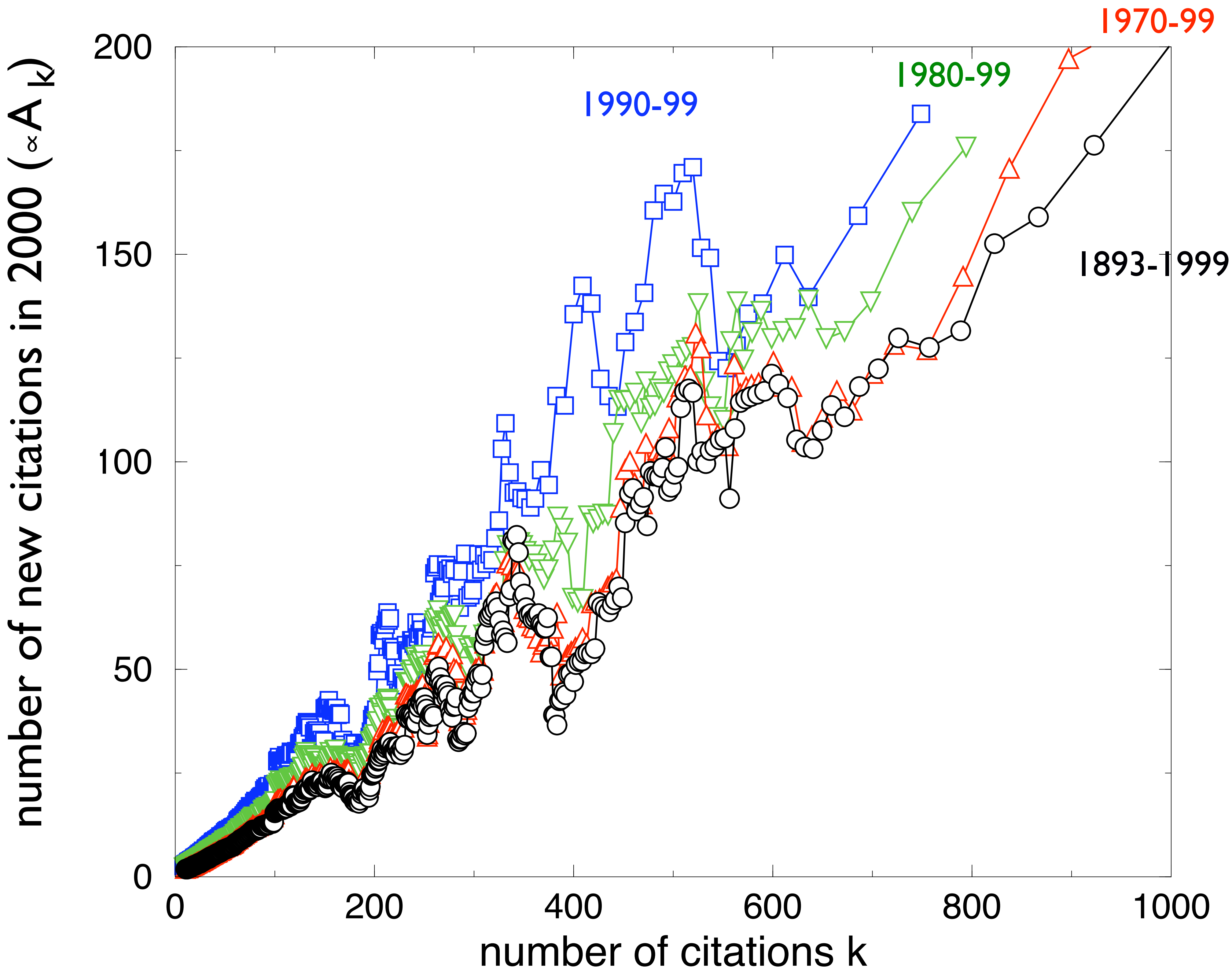
Distribution of Citations of Physical Review



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Formulation & analysis of realistic social network models and their dynamics (*communities, frustration, etc*.).