Scaling functions for nonequilibrium fluctuations: A picture gallery

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The emergence of non-gaussian distributions for macroscopic quantities in nonequilibrium steady states is discussed with emphasis on the effective criticality and on the ensuing universality of distribution functions. The following problems are treated in more detail: nonequilibrium interface fluctuations (the problem of upper critical dimension of the Kardar-Parisi-Zhang equation), roughness of signals displaying Gaussian 1/f power spectra (the relationship to extreme-value statistics), effects of boundary conditions (randomness of the digits of π).

Keywords: Nonequilibrium distributions, universality, scaling functions

I. INTRODUCTION

Spatially averaged (global) quantities in homogeneous equilibrium systems display Gaussian fluctuations since the correlation length is, in general, finite and the central limit theorem applies. The situation is more complicated at critical points where the long-range correlations yield diverging fluctuations which, in turn, generate nontrivial probability distribution functions (PDF-s). A remarkable simplification occurs, however, even in critical systems where the large-scale fluctuations lead to universality [1, 2]. For the PDF-s of macroscopic quantities, this means that the shape of the PDF-s depends only on a few general characteristics of the system (dimension, symmetries, range of interactions) [3, 4]. The emergence of universality from large-scale fluctuations appears to be so robust that one expects that similar mechanisms works in far from equilibrium steady states as well. Since no general theory exists for nonequilibrium systems, studying the differences between equilibrium and nonequilibrium phase transitions can indeed be instrumental in understanding some distinguishing but still robust properties of nonequilibrium systems [5, 6, 7]. In particular, it may help making inroads in the largely unknown territory of nonequilibrium PDF-s.

At first sight, universality ideas should be of restricted use since they apply only to critical points. One should remember, however, that nonequilibrium systems displaying power law behavior in their various characteristics (correlation in space or time, fluctuation power spectra, size-distributions, etc.) are abundant in nature. Examples range from interface fluctuations [8] and dissipation in turbulent systems [9] to voltage fluctuations in resistors [10], and to the number of earthquakes vs. their magnitude [11]. The underlying reason for the ubiquity of power-laws is not understood (the widely used expression *self-organized criticality* [12] is a testimony for this fact) but the observed effective criticality suggests that a classification of nonequilibrium PDF-s can be developed using the logics of critical phenomena. Namely, strong fluctuations and power-law correlations imply universal scaling functions for the PDF-s and, consequently, the nonequilibrium PDF-s in a large number of phenomena can be determined by studying the nonequilibrium universality classes.

Compared with equilibrium systems, complications are expected to arise from the fact that the properties of nonequilibrium steady states are determined not only by the interactions but by the dynamics as well. Thus, dynamical symmetries (conservation laws, the effects of breaking of time-reversal symmetry, etc.) should also play an important role in the classification. Furthermore, in spite of being universal, the scaling functions do depend on the boundary conditions [13]. Although this is an extra complication, it indicates that the scaling functions may be suitable for describing an important feature of nonequilibrium states, namely, that the bulk behavior depends on the boundary conditions (note that fluxes are often generated by an appropriate preparation of the boundaries).

General (field-theoretic) studies of nonequilibrium universality classes [5, 14, 15] do not address the question of distribution functions due to technical difficulties. For practical purposes, on the other hand, it is important to have the scaling functions associated with the PDF-s since, as we shall see below, they provide a possibility for "fit-free" comparisons with experiments. Once a gallery of such scaling functions has been built, it can be used to identify symmetries and underlying dynamical mechanisms in experimental systems; to discover analogies between seemingly different systems due to both belonging to the same universality class, and the applications are restricted only by imagination.

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The project of building the picture gallery of scaling functions associated with nonequilibrium PDF-s has been going on (perhaps unknowingly) for several years. For simple systems, the scaling functions have been found analytically [16, 17, 18, 19, 20, 21, 22]. In most of the cases, however, the PDF of the "microscopic" configurations is unknown and thus one has to resort to other means of calculations. One of them is a phenomenological approach [23, 24, 25] which consists of introducing an effective Gaussian action with singular dispersion and fixing the dispersion to yield the observed scale-invariant fluctuations. The other is the brute force simulations of models which are believed to be in the same universality class as the system at hand [26, 27]. The resulting picture gallery is far from complete but contains a few interesting pieces which will be presented below. First, I will show how the scaling functions are defined and calculated using simple examples from surface growth problems (Sec.II). The details of the derivation will be demonstrated on the example of a one dimensional ($d = 1$) surface dynamics that is equivalent to the problem of Gaussian $1/f$ noise (Sec.III C). As we shall see, this calculation establishes a connection between the the $1/f$ noise and one of the limiting distributions of extreme statistics. Applications of the scaling functions will be discussed in Sec.III B with details presented in connection with the problem of the upper critical dimension of the Kardar-Parisi-Zhang (KPZ) equation. Finally, as a demonstration of the importance of the boundary conditions, we shall discuss the problem of the randomness of the digits of π (Sec.IV).

II. SURFACE GROWTH AND SCALING FUNCTIONS

Among the nonequilibrium systems displaying "effective" criticality, growing surfaces provide a conceptually simple and versatile laboratory from both experimental and theoretical point of view[8, 28]. The criticality here means that growing surfaces are usually rough i.e. the mean-square fluctuations of the interface diverge with system size. More

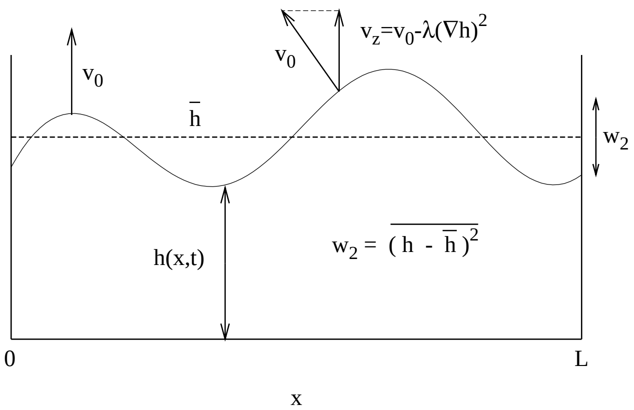

FIG. 1: Surface formed by deposition and evaporation. The vertical velocity of the surface $v_z = \partial_t h(x,t)$ is assumed to depend on the local properties, $v_z(\partial_x h, \partial_x^2 h, ...)$ and, at a macroscopic level of description, only a low-order gradient expansion of v_z is kept. The growth models differ from each other by the physical processes which are considered to be relevant and thus by keeping only the appropriate terms in the gradient expansion.

precisely, let us denote the height of the surface above a d dimensional substrate of characteristic linear dimension L by $h(\vec{r},t)$ (see Fig.1). Then the mean-square fluctuations are defined as

$$w_2 = \frac{1}{A_L}\sum_{\vec{r}} \left[\, h(\vec{r},t) - \overline{h}\,\right]^2 \tag{1}$$

where the spatially averaged height is given by $\overline{h} = \sum_{\vec{r}} h(\vec{r},t)/A_L$ with A_L being the area of the substrate. For rough surfaces one finds that the steady state average of the fluctuations $\langle w_2 \rangle_L$ diverges as $\langle w_2 \rangle_L \sim L^{2\chi}$, and the critical exponent χ is, in principle, a characteristic of the universality class the growth process belongs to.

The divergence of $\langle w_2 \rangle_L$ suggests that the width (or roughness) distribution $P_L(w_2)dw_2$, defined as the probability that w_2 is in the interval $[w_2, w_2 + dw_2]$, is a natural choice when searching for nontrivial distributions. Indeed, if the picture about criticality is correct then $\langle w_2 \rangle_L$ gives the only relevant scale in the problem and thus it follows from dimensional analysis that $P_L(w_2)$ can be expressed in the following form

$$P_L(w_2) \approx \frac{1}{\langle w_2 \rangle_L}\Phi\left(\frac{w_2}{\langle w_2 \rangle_L}\right) \tag{2}$$

where $\Phi(x)$ is a universal scaling function, the object of our main interest in this talk.

From a utilitarian point of view, three questions should be immediately answered. First, can this quantity be measured in experiments; second, can we calculate it theoretically; and third, are the $\Phi(x)$-s of different universality classes sufficiently different to be distinguishable. The answer to all three questions is a yes. Present day experiments can measure surfaces at high resolution [29, 30] and thus the $P(w_2)$ distribution can be built [23]. As to the theoretical calculation, if a model can be simulated then $P(w_2)$ can, of course, be measured. In simple cases, however, one may know the nonequilibrium steady-state distribution $\mathcal{P}[h(\vec{r})] \sim \exp\{-S[h]\}$ and Φ can be calculated exactly (an example will be presented in Sec.III C). Here we sketch only the first steps of the calculation.

Formally, $P(w_2)$ is obtained from $\mathcal{P}[h(\vec{r})]$ as a path integral

$$P(w_2) = \int \mathcal{D}h(\vec{r})\, \delta\left(w_2 - [\,\overline{h^2} - \overline{h}^2\,]\right) \exp\{-S[h]\}\ . \tag{3}$$

where the overbar $\overline{h^n}$ denotes spatial averaging. In practice, it is easier to calculate the generating function

$$G(s) = \int_0^\infty e^{-sw_2} P(w_2) dw_2 = \mathcal{N} \int \mathcal{D}h(\vec{r})\ \exp\{-S[h] - s[\,\overline{h^2} - \overline{h}^2\,]\}\ , \tag{4}$$

where $\mathcal{N}$ is a normalization constant. The above expression is instrumental in finding models where $P(w_2)$ can be obtained analytically. The path integral in Eq.(4) is the partition function of a model with an effective action $S_{eff}[h] = S[h] + s[\,\overline{h^2} - \overline{h}^2\,]$. The terms added to $S[h]$ are quadratic functionals of h and so one expects that the generating function and thus $P(w_2)$ can be evaluated exactly if the original model defined by $S[h]$ is solvable. There

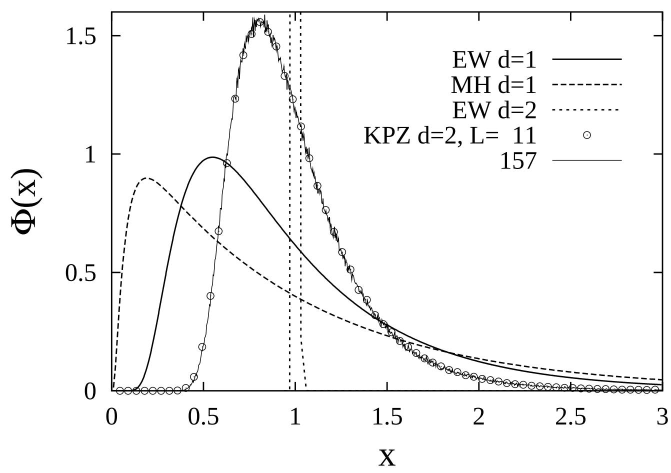

FIG. 2: Fluctuation distributions for the $d = 1$ dimensional Edwards-Wilkinson (EW) and Mullins-Herring (MH) models, and for the $d = 2$ EW and Kardar-Parisi-Zhang (KPZ) models. The scaled probability of the roughness $\Phi = \langle w_2\rangle P(w_2)$ is plotted against the scaled roughness $x = w_2/\langle w_2\rangle$. The $d = 1$ scaling functions are the limiting functions when the size of the system L goes to infinity. For the $d = 2$ EW model, a large but finite-L scaling function is plotted since one obtains a delta function in the $L \to \infty$ limit. The KPZ distributions were built by simulating $L \times L$ systems.

are several growth models which are exactly solvable since $S[h]$ is a quadratic functional of h. A notable example is the Edwards-Wilkinson (EW) model [8] describing growth governed by surface tension and having $S[h] \sim \overline{(\nabla h)^2}$. Another example is the Mullins-Herring (MH) equation [8] modeling curvature driven growth and having a steady state characterized by $S[h] \sim \overline{(\Delta h)^2}$. These models belong to distinct universality classes since $\langle w_2\rangle_L \sim L^{2\chi}$ with $\chi_{EW,d=1} = 1/2$ while $\chi_{MH,d=1} = 1$ in $d = 1$ dimension. Accordingly, their scaling functions should be different. The $d = 1$ scaling functions for these models [16, 17, 19, 23] are displayed on Fig.2 and one can see that they are easily distinguishable.

Fig.2 also shows the Φ-s for the $d = 2$ EW and KPZ models [19, 23, 26] (the latter is a nonlinear model discussed in Sec.III B) which are again in different universality classes since $\chi_{EW,d=2} = 0$ (logarithmic divergence) while $\chi_{KPZ,d=2} \approx 0.39$. As can be seen from Fig.2 the scaling functions strongly differ from each other and, furthermore, they are also distinct from the Φ-s of the $d = 1$ models. Thus Fig.2 gives an "answer by example" to the third utilitarian question posed above.

We conclude this section with two notes on Fig.2. First, it should be remarked that the $d = 2$ EW scaling function approaches a delta function on the scale of $w_2/\langle w_2\rangle$, meaning that the fluctuations of w_2 do not diverge in the $L \to \infty$

limit. The delta function, however, hides an interesting structure which can be revealed by an appropriate choice of the scaling variables [19]. Second, the collapse of the $L = 11$ and $L = 157$ KPZ results demonstrates an important point about the explicite L-dependence of the Φ-s. Indeed, the scaling functions have finite size corrections, i.e. they have explicite L-dependence in addition to the L-dependence through the argument $x = w_2/\langle w_2\rangle$. Not much is known about the finite-size correction of Φ-s but the experience with a large number of models indicate that the explicite L-dependence is negligible when the number of surface sites becomes smaller than the number of sites in the bulk. This is what we see in Fig.2 on the example of the $d = 2$ KPZ model.

III. APPLICATIONS

A. Discovering structures and similarities in experiments and in simulations

A straightforward way of applying the picture gallery built for surface models is to take the results of a surface growth experiment, build the scaling function of the width distribution, and compare it with the pictures in the gallery. If one finds agreement with one of the pictures then one can reason about the physical processes which are relevant in the given growth process. These types of arguments can be found for example in Ref.[23].

A more sophisticated application was the establishment of a connection between the dissipation fluctuations in a turbulence experiment and the magnetization fluctuations in the $d = 2$ XY model at low temperatures [31]. Since the low-temperature fluctuations in the $d = 2$ XY model are equivalent to the surface fluctuations $d = 2$ Edwards-Wilkinson model, the discovery of the above connection prompted a search for an interface interpretation of the dissipative structures in the turbulent system [32].

In more theoretical applications, the scaling functions developed for surfaces have been used to find the universality class of massively parallel algorithms and thus to establish their scalability [33]. Furthermore, they were also instrumental in establishing the universality class of fronts propagating into unstable phases [34]. Below, we describe in detail two more theoretical applications. The first is notable for its logics of approaching a controversial problem while the second is remarkable for the puzzle in the end result.

B. Upper critical dimension of the Kardar-Parisi-Zhang model

A nontrivial feature of the approach advocated in this talk is that the building of the PDF-s does not involve any fitting or other procedures involving subjective judgment. In order to appreciate the advantages of this feature, we shall discuss below the problem of upper critical dimension of the KPZ equation.

The KPZ equation [35] is the simplest nonlinear generalization of the EW model. In addition to the surface-tension effects, it also takes into account that the surface grows along its normal provided the attachment dynamics is isotropic. Then, as one can see from Fig.1, the vertical velocity of the surface has a correction term proportional to $(\nabla h)^2$, and the equation, in lowest order in the nonlinearities, can be written as

$$\partial_t h = \nu \vec{\nabla}^2 h + \lambda(\vec{\nabla} h)^2 + \eta\,. \tag{5}$$

Here ν and λ are parameters, while $\eta(\vec{r}, t)$ is a Gaussian white noise. The above equation has been investigated intensively since it gives account of a number of interesting phenomena (Burgers turbulence, directed polymers in random media, etc.). Nevertheless, a number of issues remained unsolved. In particular, there is no agreement on upper critical dimension (d_u) above which a mean-field theory would be valid. Mode-coupling and other phenomenological theories (see Ref.[36] and references therein) suggest that $d_u = 4$ while all the numerical work (see e.g. Ref.[37]) fail to find a finite d_u. There are, of course, problems with the approximations in the phenomenological theories as well as with the multiparameter fits of the simulation results, and the debate is rather controversial. We show now, following the lines of our recent work [26], how the scaling functions of the roughness can shed some light on this controversy.

The logics of the solution originates in the theory of critical phenomena. It is known that the universal scaling functions depend on d for $d \le d_u$ while their dependence is lost in the mean-field region ($d > d_u$). Thus finding smoothly varying Φ-s in dimensions $d-1, d, d+1$ should lead to the conclusion that $d \ne d_u$. To use this argument, we compare on Fig.3 the $d = 3, 4, 5$ KPZ scaling functions using simulation data for restricted solid-on-solid (RSOS) growth models [26, 37]. The message from Fig.3 is clear: It is highly unlikely that $d_u = 4$ would be the upper critical dimension of the KPZ equation. Note that we arrived at this result without using any approximation or fitting procedure. The only way out of the conclusion $d \ne d_u$ would be if there would be some finite-size corrections to the scaling functions which persist at large sizes. One cannot see such corrections for $d = 2$ in Fig.2 and the finite-size analysis of the Φ-s in $3 \le d \le 5$ leads to a similar conclusion [26].

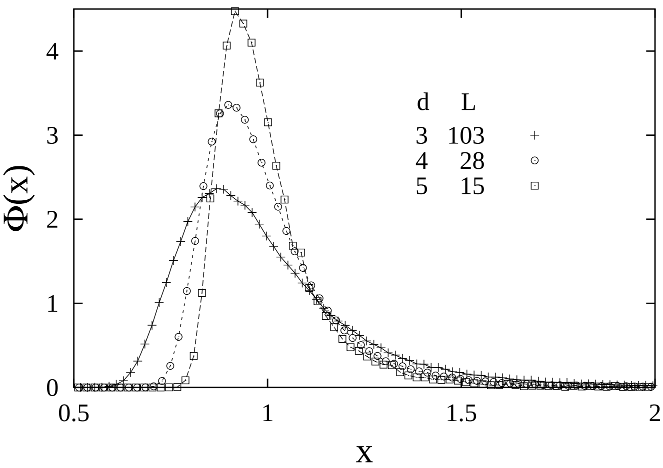

FIG. 3: Scaling functions for the roughness distributions in the $d = 3-5$ dimensional KPZ models. The scaled probability of the roughness $\Phi = \langle w_2\rangle P(w_2)$ is plotted against $x = w_2/\langle w_2\rangle$. The characteristic linear size of the simulated systems is given by L.

C. $1/f$ noise and the Fisher-Tippett-Gumbel distribution

Studies of the scaling functions often yield connections which are intriguing but hard to understand. The aim of this section is to derive an example of such puzzling result. Namely, we shall show that the PDF of the mean-square fluctuations in $1/f$ noise is related to the Fisher-Tippett-Gumbel distribution which is one of the limiting distributions of extreme value statistics. The derivation in itself is worth going through since it demonstrates the steps of calculating $P(w_2)$ in a case where the usual choice of the scaling variable $w_2/\langle w_2\rangle$ would yield a delta-function PDF in the large-size limit. The derivation is an expanded version of what can be found in a recent letter of ours [20].

$1/f$ noise is usually associated with a time signal so we should start by imagining the interface in Fig.1 as a time signal i.e. we should make the replacements $x \to t$ and $L \to T$ with T being the period of the signal. The mean square fluctuations $w_2(h) = \overline{[\,h(t) - \overline{h}\,]^2}$ are given by averaging over period $[0, T]$. Then the expressions for $P(w_2)$ and for the generating function $G(s)$ [Eqs.(3) and (4)] are unchanged and one is left with the problem of finding a suitable $S[h]$ representing the $1/f$ noise. In Ref.[20], we proposed that the path probability of a Gaussian, periodic, random phase, and perfectly $1/f$ noise with the dispersion being linear for all frequencies, can be described by the following action

$$S = \sigma \sum_{n=-N}^{N} |n||c_n|^2 = 2\sigma \sum_{n=1}^{N} n|c_n|^2 \,, \tag{6}$$

where σ is an effective surface tension in the language of surfaces and the c_n-s are the Fourier coefficients of the periodic [$h(t+T) = h(t)$] signal

$$h(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t/T} \,, \qquad c_{-n} = c_n^\star \,. \tag{7}$$

Several notes are in order here. First, there is a cutoff in above Fourier series. Its meaning is that the time signal is resolved at time increments of $\tau = T/N$. Using N finite makes the steps of the calculation simple and the $N \to \infty$ (equivalent to the $T \to \infty$ limit) can be taken in the final results. Second, the power spectrum with the above action is indeed $1/f$

$$\langle |c_n|^2\rangle \sim 1/|n| \,, \tag{8}$$

and, third, there is a simple meaning to w_2 in the noise terminology since

$$w_2 = 2\sum_{n=1}^{N} |c_n|^2 \,, \tag{9}$$

i.e. w_2 is the integrated power spectrum.

Turning now to the evaluation of $P(w_2)$, let us note that the functional integral (4) can be written in terms of the Fourier amplitudes as

$$G(s) = \bar{\mathcal{N}} \prod_{n=1}^{N} \int_{-\infty}^{\infty} dc_n \int_{-\infty}^{\infty} dc_n^* \exp\left[-\sum_{m=1}^{N} 2(\sigma m + s)|c_m|^2 \right] , \tag{10}$$

where $\bar{\mathcal{N}}$ is a constant to be determined from the normalization ($G(0) = 1$) condition. Carrying out the integrals, one finds the generating function in terms of a product

$$G(s) = \prod_{n=1}^{N} \frac{\sigma n}{\sigma n + s} . \tag{11}$$

The structure of the above expression is common to various Gaussian growth models where $S[h]$ is a quadratic functional of h. For example, to obtain the $G(s)$ for the EW model, one should just make the substitution $n \to n^2$ and $\sigma \to \sigma/N$. Since $G(s)$ has poles on the negative real axis, there is a straighforward method for calculating $P(w_2)$. Namely, the inverse Laplace transform of $G(s)$ is an integral along the imaginary axis,

$$P(w_2) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{w_2 s} \prod_{n=1}^{N} \frac{\sigma n}{\sigma n + s} , \tag{12}$$

and thus $P(w_2)$ is obtained as a sum of contributions from the poles. The method works [16, 17, 23] but in our case it yields a rather complicated expression which can be shown to converge to a delta function if the usual scaling variable $x = w_2/\langle w_2 \rangle$ is used. Actually, this result can be understood by just calculating the first two cumulants of w_2. The first cumulant diverges for $N \to \infty$ as

$$\langle w_2 \rangle = -\frac{dG}{ds}\bigg|_{s=0} = \frac{1}{\sigma} \sum_{n=1}^{N} n^{-1} \approx \frac{1}{\sigma} \left[\ln N + \gamma \right] \tag{13}$$

where $\gamma = 0.577...$ is the Euler constant, while the fluctuations of w_2 are finite

$$\langle w_2^2 \rangle - \langle w_2 \rangle^2 = \frac{a^2}{\sigma^2} ; \qquad a = \frac{\pi}{\sqrt{6}} . \tag{14}$$

This means that if we scale w_2 with $\langle w_2 \rangle$ then the width of the distribution goes to zero for $N \to \infty$ hence the conclusion about the delta function. In order not to lose information about the possible structure at small $w_2 - \langle w_2 \rangle$, one should introduce a scaling variable which expands the delta function. This can be achieved by introducing the following variable [19]

$$y = \frac{w_2 - \langle w_2 \rangle}{\sqrt{\langle w_2^2 \rangle - \langle w_2 \rangle^2}} . \tag{15}$$

Substituting w_2 from the above expression into Eq.(12) and using Eqs.(13) and (14), one finds that the limit $N \to \infty$ can be taken and a scaling function in terms of y emerges

$$\Phi(y) \equiv \sqrt{\langle w_2^2 \rangle - \langle w_2 \rangle^2} P(w_2) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{xs} \prod_{n=1}^{\infty} \frac{e^{\frac{s}{an}}}{1 + \frac{s}{an}} . \tag{16}$$

Noting that infinite product above can be expressed [38] through the Γ function

$$\prod_{n=1}^{\infty} \frac{e^{\frac{s}{an}}}{1 + \frac{s}{an}} = e^{\gamma s/a} \Gamma\left(1 + \frac{s}{a}\right) , \tag{17}$$

and using Euler's integral representation for the Γ function, one finds a surprisingly simple analytical result

$$\Phi(y) = a e^{-(ay+\gamma) - e^{-(ay+\gamma)}} . \tag{18}$$

The real surprise actually is that $\Phi(y)$, shown in Fig. 4, is the the so called Fisher-Tippett-Gumbel distribution [39] which is one of the three limiting forms of extreme value statistics.

It should be emphasized that we do not see any physical reason why should there be a connection between the fluctuations of a $1/f$ signal and the extreme value distributions. It seems to be a puzzle, however, whose solution may reveal an interesting picture.

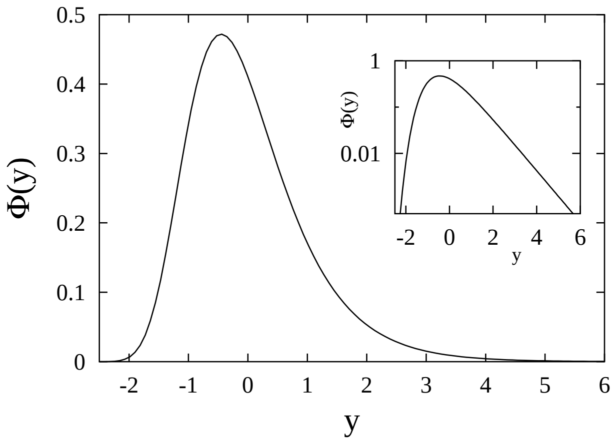

FIG. 4: The Fisher-Tippett-Gumbel distribution. The inset shows it on semilogarithmic scale to demonstrate the exponential decay at large arguments.

IV. EFFECT OF BOUNDARY CONDITIONS

As a final topic, I would like to consider the boundary-condition dependence of the scaling functions. As discussed in previous sections, the universality is due to the diverging fluctuations which in turn imply that the correlation length is infinite in the system. The infinite correlations mean that the boundaries are felt in the bulk and thus it is not entirely unexpected that the scaling functions are sensitive to the boundaries. The important question is how large the boundary effects are.

The effects of boundaries have been studied [21] for Gaussian signals with $1/f^\alpha$ power spectrum. It was found that the boundary effects are large for $\alpha > 4$, easily noticeable in the range $1 < \alpha < 4$, and they disappear entirely for $\alpha < 0.5$. In order to demonstrate the magnitude of the boundary effect in the physically most relevant range of $1 < \alpha < 4$, I would like to consider the problem of the randomness of the digits of π which, as we shall see, corresponds to an $\alpha = 2$ problem.

The statistical properties of the digits of π appears to occupy the mind of a few mathematicians [40] [1]. Although no rigorous proof exists yet, it is believed that the digits are random. This belief is based on generating and analyzing a large number ($10^7 - 10^8$) of digits. We shall test the belief of randomness by mapping the digits onto an interface (see Fig.5) and calculating the scaling function of the width of the interface.

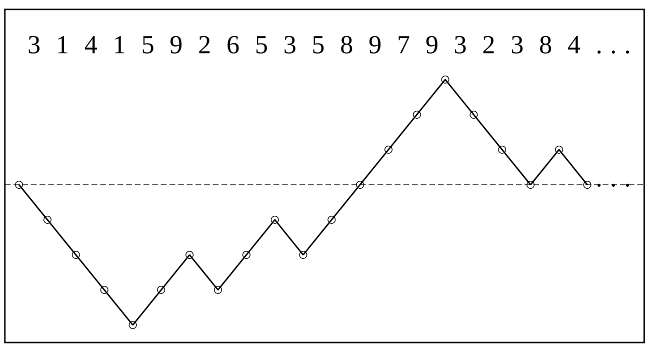

FIG. 5: Mapping between the digits of π and a configuration of the so called rooftop model [41, 42] of interface. The slope of the interface can be only ± 1 with -1 occurring for digits $0 - 4$ and $+1$ for $5 - 9$.

If the digits of π are random then the interface shown in Fig.5 is just a random walk equivalent to the steady state of the Edwards-Wilkinson model as well as to Gaussian signals with $1/f^2$ power spectrum. For periodic $1/f^2$ signals

[1] See e.g. http://www.sfu.ca/˜ pborwein

the width distribution was calculated in Ref. [16]. It is not straightforward, however, to make a comparison with this distribution function since the digits of π provide only a long but single signal (not unlike to some experimental situations). A histogram of roughness can be built, nevertheless, by calculating w_2 for segments (windows) of the signal whose size is much smaller than the total length but at the same time large enough that the finite-size effects in the scaling function would be small. Clearly, the boundary conditions for the segments are not periodic (we call them window boundary conditions, WBC). The PDF of the roughness using WBC turns out to be a well defined function in the limit of the total length of the signal going to infinity, and, furthermore, this function is found [21] to be independent of the boundary conditions for the full signal. Comparison of this function with the width distribution of π is displayed on Fig.6 where one can also see the scaling function for periodic boundary conditions (PBC). The difference between the PBC and WBC scaling functions is easily observable and it is clear that we would have concluded incorrectly about the non randomness of the digits of π had we ignored the boundary condition dependence of the scaling functions.

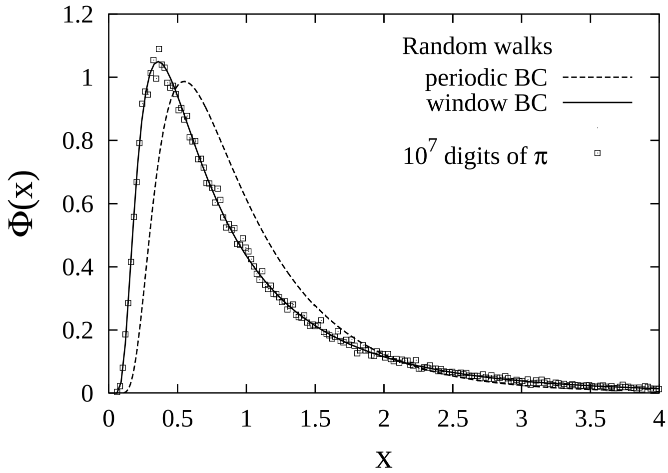

FIG. 6: Distribution of the roughness of π calculated by mapping its first 10^7 digits onto the rooftop interface model (Fig.5). The scaled distribution $\Phi =< w_2 > P(w_2)$ is obtained by using a measuring window of 10^3 digits and the scaled variable is $x = w_2/ < w_2 >$. The comparison is with random walks with periodic- and window (free) boundary conditions. The collapse with the window BC curve is in agreement with the notion that the digits of π are random.

The moral from the story of π is twofold. On one hand it cautions us that the scaling functions are useful theoretical instruments only if the boundary conditions for experimental data are carefully specified both at the measuring and at the analyzing stage. On the other hand, the boundary-condition dependence of the scaling functions suggests that they "*hear the shape of the drum*" i.e. they may be used to see the shape of objects which are embedded in other media and can be seen only through their fluctuations.

V. FINAL REMARKS

It should be clear that only a small part of the picture gallery was exhibited in this lecture. The gallery is far from complete and we expect that scaling functions originating from newly discovered nonequilibrium universality classes will enrich it regularly. Furthermore, we expect that limiting distributions such as the ones emerging in extreme statistics will also be included and will have much wider use in physics. Finally, strongly fluctuating quantum systems may also provide valuable novelties.

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